If we choose δ' close enough to δ, the quantity $\left(\frac{1}{\lambda}+\frac{1}{\delta'}\right)$ will be negative, and we end up with

$$\psi(hu_1) < 0 \qquad \text{for} \quad h \neq 0 \quad \text{small} \ . \tag{12}$$

On the other hand, we check directly that $\psi(0)=0$. This shows that 0 cannot be a minimizer of ψ, not even a local one. So $\overline{u} \neq 0$ and $\overline{u} \neq \Lambda_o^{-1}(0) = 0$. □

Corollary 2. *Assume H is C^2 and (a_∞, b_∞)-subquadratic at infinity. Let ξ_1, $\ldots, \xi_N$ be the equilibria, that is, the solutions of $H'(\xi)=0$. Denote by ω_k the smallest eigenvalue of $H''(\xi_k)$, and set:*

$$\omega := \mathrm{Min}\ \{\omega_1, \ldots, \omega_k\} \ . \tag{13}$$

If:

$$\frac{T}{2\pi} b_\infty < -E\left[-\frac{T}{2\pi} a_\infty\right] < \frac{T}{2\pi}\omega \tag{14}$$

then minimization of ψ yields a non-constant T-periodic solution $\overline{x}$.

We recall once more that by the integer part $E[\alpha]$ of $\alpha \in \mathbb{R}$, we mean the $a \in \mathbb{Z}$ such that $a < \alpha \leq a+1$. For instance, if we take $a_\infty = 0$, Corollary 2 tells us that $\overline{x}$ exists and is non-constant provided that:

$$\frac{T}{2\pi} b_\infty < 1 < \frac{T}{2\pi} \tag{15}$$

or

$$T \in \left(\frac{2\pi}{\omega}, \frac{2\pi}{b_\infty}\right) \ . \tag{16}$$

Proof. The spectrum of Λ is $\frac{2\pi}{T}\mathbb{Z} + a_\infty$. The largest negative eigenvalue λ is given by $\frac{2\pi}{T}k_o + a_\infty$, where

$$\frac{2\pi}{T}k_o + a_\infty < 0 \leq \frac{2\pi}{T}(k_o+1) + a_\infty \ . \tag{17}$$

Hence:

$$k_o = E\left[-\frac{T}{2\pi}a_\infty\right] \ . \tag{18}$$

The condition $\gamma < -\lambda < \delta$ now becomes:

$$b_\infty - a_\infty < -\frac{2\pi}{T}k_o - a_\infty < \omega - a_\infty \tag{19}$$

which is precisely condition (14). □

Lemma 3. *Assume that H is C^2 on $\mathbb{R}^{2n}\backslash\{0\}$ and that $H''(x)$ is non-degenerate for any $x \neq 0$. Then any local minimizer $\widetilde{x}$ of ψ has minimal period T.*