Hamiltonian Mechanics

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Abstract. The abstract should summarize the contents of the paper using at least 70 and at most 150 words. It will be set in 9-point font size and be inset 1.0 cm from the right and left margins. There will be two blank lines before and after the Abstract. . . .

1 Fixed-Period Problems: The Sublinear Case

With this chapter, the preliminaries are over, and we begin the search for periodic solutions to Hamiltonian systems. All this will be done in the convex case; that is, we shall study the boundary-value problem

$$\begin{aligned} \dot{x} &= JH'(t,x) \\ x(0) &= x(T) \end{aligned}$$

with $H(t,\cdot)$ a convex function of x, going to $+\infty$ when $\|x\| \to \infty$.

1.1 Autonomous Systems

In this section, we will consider the case when the Hamiltonian $H(x)$ is autonomous. For the sake of simplicity, we shall also assume that it is C^1.

We shall first consider the question of nontriviality, within the general framework of (A_∞, B_∞)-subquadratic Hamiltonians. In the second subsection, we shall look into the special case when H is $(0, b_\infty)$-subquadratic, and we shall try to derive additional information.

The General Case: Nontriviality. We assume that H is (A_∞, B_∞)-subquadratic at infinity, for some constant symmetric matrices A_∞ and B_∞, with $B_\infty - A_\infty$ positive definite. Set:

$$\gamma := \text{smallest eigenvalue of} \quad B_\infty - A_\infty \tag{1}$$

$$\lambda := \text{largest negative eigenvalue of} \quad J\frac{d}{dt} + A_\infty \ . \tag{2}$$

Theorem 21 tells us that if $\lambda + \gamma < 0$, the boundary-value problem:

$$\begin{aligned} \dot{x} &= JH'(x) \\ x(0) &= x(T) \end{aligned} \tag{3}$$

has at least one solution $\overline{x}$, which is found by minimizing the dual action functional:

$$\psi(u) = \int_o^T \left[\frac{1}{2}\left(\Lambda_o^{-1}u,u\right) + N^*(-u)\right] dt \tag{4}$$

on the range of Λ, which is a subspace $R(\Lambda)_L^2$ with finite codimension. Here

$$N(x) := H(x) - \frac{1}{2}\left(A_\infty x, x\right) \tag{5}$$

is a convex function, and

$$N(x) \le \frac{1}{2}\left(\left(B_\infty - A_\infty\right)x, x\right) + c \quad \forall x \ . \tag{6}$$

Proposition 1. *Assume $H'(0) = 0$ and $H(0) = 0$. Set:*

$$\delta := \liminf_{x \to 0} 2N(x)\left\|x\right\|^{-2} \ . \tag{7}$$

If $\gamma < -\lambda < \delta$, the solution $\overline{u}$ is non-zero:

$$\overline{x}(t) \ne 0 \quad \forall t \ . \tag{8}$$

Proof. Condition (7) means that, for every $\delta' > \delta$, there is some $\varepsilon > 0$ such that

$$\left\|x\right\| \le \varepsilon \Rightarrow N(x) \le \frac{\delta'}{2}\left\|x\right\|^2 \ . \tag{9}$$

It is an exercise in convex analysis, into which we shall not go, to show that this implies that there is an $\eta > 0$ such that

$$f\left\|x\right\| \le \eta \Rightarrow N^*(y) \le \frac{1}{2\delta'}\left\|y\right\|^2 \ . \tag{10}$$

Fig. 1. This is the caption of the figure displaying a white eagle and a white horse on a snow field

Since u_1 is a smooth function, we will have $\left\|hu_1\right\|_\infty \le \eta$ for h small enough, and inequality (10) will hold, yielding thereby:

$$\psi(hu_1) \le \frac{h^2}{2}\frac{1}{\lambda}\left\|u_1\right\|_2^2 + \frac{h^2}{2}\frac{1}{\delta'}\left\|u_1\right\|^2 \ . \tag{11}$$

If we choose δ' close enough to δ, the quantity $\left(\frac{1}{\lambda}+\frac{1}{\delta'}\right)$ will be negative, and we end up with

$$\psi(hu_1) < 0 \qquad \text{for} \quad h \neq 0 \quad \text{small} \ . \tag{12}$$

On the other hand, we check directly that $\psi(0) = 0$. This shows that 0 cannot be a minimizer of ψ, not even a local one. So $\overline{u} \neq 0$ and $\overline{u} \neq \Lambda_o^{-1}(0) = 0$. □

Corollary 2. *Assume H is C^2 and (a_∞, b_∞)-subquadratic at infinity. Let ξ_1, $\dots, \xi_N$ be the equilibria, that is, the solutions of $H'(\xi) = 0$. Denote by ω_k the smallest eigenvalue of $H''(\xi_k)$, and set:*

$$\omega := \mathrm{Min}\ \{\omega_1, \dots, \omega_k\} \ . \tag{13}$$

If:

$$\frac{T}{2\pi} b_\infty < -E\left[-\frac{T}{2\pi} a_\infty\right] < \frac{T}{2\pi}\omega \tag{14}$$

then minimization of ψ yields a non-constant T-periodic solution $\overline{x}$.

We recall once more that by the integer part $E[\alpha]$ of $\alpha \in \mathbb{R}$, we mean the $a \in \mathbb{Z}$ such that $a < \alpha \leq a + 1$. For instance, if we take $a_\infty = 0$, Corollary 2 tells us that $\overline{x}$ exists and is non-constant provided that:

$$\frac{T}{2\pi} b_\infty < 1 < \frac{T}{2\pi} \tag{15}$$

or

$$T \in \left(\frac{2\pi}{\omega}, \frac{2\pi}{b_\infty}\right) \ . \tag{16}$$

Proof. The spectrum of Λ is $\frac{2\pi}{T}\mathbb{Z} + a_\infty$. The largest negative eigenvalue λ is given by $\frac{2\pi}{T}k_o + a_\infty$, where

$$\frac{2\pi}{T}k_o + a_\infty < 0 \leq \frac{2\pi}{T}(k_o + 1) + a_\infty \ . \tag{17}$$

Hence:

$$k_o = E\left[-\frac{T}{2\pi}a_\infty\right] \ . \tag{18}$$

The condition $\gamma < -\lambda < \delta$ now becomes:

$$b_\infty - a_\infty < -\frac{2\pi}{T}k_o - a_\infty < \omega - a_\infty \tag{19}$$

which is precisely condition (14). □

Lemma 3. *Assume that H is C^2 on $\mathbb{R}^{2n} \backslash \{0\}$ and that $H''(x)$ is non-degenerate for any $x \neq 0$. Then any local minimizer $\widetilde{x}$ of ψ has minimal period T.*

Proof. We know that $\widetilde{x}$, or $\widetilde{x}+\xi$ for some constant $\xi \in \mathbb{R}^{2n}$, is a T-periodic solution of the Hamiltonian system:

$$\dot{x} = JH'(x) \ . \tag{20}$$

There is no loss of generality in taking $\xi = 0$. So $\psi(x) \geq \psi(\widetilde{x})$ for all $\widetilde{x}$ in some neighbourhood of x in $W^{1,2}\left(\mathbb{R}/T\mathbb{Z};\mathbb{R}^{2n}\right)$.

But this index is precisely the index $i_T(\widetilde{x})$ of the T-periodic solution $\widetilde{x}$ over the interval $(0,T)$, as defined in Sect. 2.6. So

$$i_T(\widetilde{x}) = 0 \ . \tag{21}$$

Now if $\widetilde{x}$ has a lower period, T/k say, we would have, by Corollary 31:

$$i_T(\widetilde{x}) = i_{kT/k}(\widetilde{x}) \geq k i_{T/k}(\widetilde{x}) + k - 1 \geq k - 1 \geq 1 \ . \tag{22}$$

This would contradict (21), and thus cannot happen. □

Notes and Comments. The results in this section are a refined version of [1]; the minimality result of Proposition 14 was the first of its kind.

To understand the nontriviality conditions, such as the one in formula (16), one may think of a one-parameter family x_T, $T \in \left(2\pi\omega^{-1}, 2\pi b_\infty^{-1}\right)$ of periodic solutions, $x_T(0) = x_T(T)$, with x_T going away to infinity when $T \to 2\pi\omega^{-1}$, which is the period of the linearized system at 0.

Table 1. This is the example table taken out of *The TEXbook,* p. 246

Year	World population
8000 B.C.	5,000,000
50 A.D.	200,000,000
1650 A.D.	500,000,000
1945 A.D.	2,300,000,000
1980 A.D.	4,400,000,000

Theorem 4 (Ghoussoub-Preiss). *Assume $H(t,x)$ is $(0,\varepsilon)$-subquadratic at infinity for all $\varepsilon > 0$, and T-periodic in t*

$$H(t,\cdot) \qquad \text{is convex} \quad \forall t \tag{23}$$

$$H(\cdot,x) \qquad \text{is} \quad T\text{−periodic} \quad \forall x \tag{24}$$

$$H(t,x) \geq n\left(\|x\|\right) \qquad \text{with} \quad n(s)s^{-1} \to \infty \quad \text{as} \quad s \to \infty \tag{25}$$

$$\forall \varepsilon > 0 \ , \qquad \exists c \ : \ H(t,x) \leq \frac{\varepsilon}{2}\left\|x\right\|^2 + c \ . \tag{26}$$

Assume also that H is C^2, and $H''(t,x)$ is positive definite everywhere. Then there is a sequence x_k, $k \in \mathbb{N}$, of kT-periodic solutions of the system

$$\dot{x} = JH'(t,x) \tag{27}$$

such that, for every $k \in \mathbb{N}$, there is some $p_o \in \mathbb{N}$ with:

$$p \geq p_o \Rightarrow x_{pk} \neq x_k \ . \tag{28}$$

□

Example 1 (External forcing). Consider the system:

$$\dot{x} = JH'(x) + f(t) \tag{29}$$

where the Hamiltonian H is $(0, b_\infty)$-subquadratic, and the forcing term is a distribution on the circle:

$$f = \frac{d}{dt}F + f_o \qquad \text{with} \quad F \in L^2\left(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{2n}\right) \ , \tag{30}$$

where $f_o := T^{-1}\int_o^T f(t)dt$. For instance,

$$f(t) = \sum_{k \in \mathbb{N}} \delta_k \xi \ , \tag{31}$$

where δ_k is the Dirac mass at $t = k$ and $\xi \in \mathbb{R}^{2n}$ is a constant, fits the prescription. This means that the system $\dot{x} = JH'(x)$ is being excited by a series of identical shocks at interval T.

Definition 5. Let $A_\infty(t)$ and $B_\infty(t)$ be symmetric operators in $\mathbb{R}^{2n}$, depending continuously on $t \in [0,T]$, such that $A_\infty(t) \leq B_\infty(t)$ for all t.

A Borelian function $H : [0,T] \times \mathbb{R}^{2n} \to \mathbb{R}$ is called (A_∞, B_∞)-*subquadratic at infinity* if there exists a function $N(t,x)$ such that:

$$H(t,x) = \frac{1}{2}\left(A_\infty(t)x, x\right) + N(t,x) \tag{32}$$

$$\forall t \ , \quad N(t,x) \qquad \text{is convex with respect to} \quad x \tag{33}$$

$$N(t,x) \geq n\left(\|x\|\right) \qquad \text{with} \quad n(s)s^{-1} \to +\infty \quad \text{as} \quad s \to +\infty \tag{34}$$

$$\exists c \in \mathbb{R} \ : \quad H(t,x) \leq \frac{1}{2}\left(B_\infty(t)x, x\right) + c \quad \forall x \ . \tag{35}$$

If $A_\infty(t) = a_\infty I$ and $B_\infty(t) = b_\infty I$, with $a_\infty \leq b_\infty \in \mathbb{R}$, we shall say that H is (a_∞, b_∞)-subquadratic at infinity. As an example, the function $\|x\|^\alpha$, with $1 \leq \alpha < 2$, is $(0, \varepsilon)$-subquadratic at infinity for every $\varepsilon > 0$. Similarly, the Hamiltonian

$$H(t,x) = \frac{1}{2}k\left\|k\right\|^2 + \left\|x\right\|^\alpha \tag{36}$$

is $(k, k+\varepsilon)$-subquadratic for every $\varepsilon > 0$. Note that, if $k < 0$, it is not convex.

Notes and Comments. The first results on subharmonics were obtained by Rabinowitz in [5], who showed the existence of infinitely many subharmonics both in the subquadratic and superquadratic case, with suitable growth conditions on H'. Again the duality approach enabled Clarke and Ekeland in [2] to treat the same problem in the convex-subquadratic case, with growth conditions on H only.

Recently, Michalek and Tarantello (see [3] and [4]) have obtained lower bound on the number of subharmonics of period kT, based on symmetry considerations and on pinching estimates, as in Sect. 5.2 of this article.

References

1. Clarke, F., Ekeland, I.: Nonlinear oscillations and boundary-value problems for Hamiltonian systems. Arch. Rat. Mech. Anal. **78** (1982) 315–333
2. Clarke, F., Ekeland, I.: Solutions périodiques, du période donnée, des équations hamiltoniennes. Note CRAS Paris **287** (1978) 1013–1015
3. Michalek, R., Tarantello, G.: Subharmonic solutions with prescribed minimal period for nonautonomous Hamiltonian systems. J. Diff. Eq. **72** (1988) 28–55
4. Tarantello, G.: Subharmonic solutions for Hamiltonian systems via a $\mathbb{Z}_p$ pseudoindex theory. Annali di Matematica Pura (to appear)
5. Rabinowitz, P.: On subharmonic solutions of a Hamiltonian system. Comm. Pure Appl. Math. **33** (1980) 609–633

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