Linear Hamiltonian Systems – Continous versus Discrete

Ondřej Došlý

Department of Mathematics, Masaryk University, Janáčkovo nám. 2a,
662 95 Brno, The Czech Republic
Email: dosly@math.muni.cz
WWW: http://www.math.muni.cz/~ndosly

Abstract. Oscillatory properties of discrete and continuous linear Hamiltonian systems are compared. In particular, possibilities to establish discrete versions of the so-called reciprocity principle and trigonometric transformation are examined.

AMS Subject Classification. 34C10, 39A10

Keywords. Hamiltonian system, reciprocity principle

1 Introduction

The aim of this contribution is to investigate relationship between oscillatory properties of the continuous (= differential) linear Hamiltonian system (further LHS)

$$(\mathrm{H}_c) \qquad x' = A(t)x + B(t)u, \quad u' = C(t)x - A^T(t)u,$$

and its difference (= discrete) counterpart

$$(\mathrm{H}_d) \qquad \Delta x_k = A_k x_{k+1} + B_k u_k, \quad \Delta u_k = C_k x_{k+1} - A_k^T u_k.$$

We suppose that $t \in I = [a, b] \subset \mathbb{R}$, $k \in [0, N] \cap \mathbb{N}$, $N \in \mathbb{N}$, both in continuous and discrete case A, B, C are $n \times n$ matrices, B, C are symmetric, i. e. $B = B^T$, $C = C^T$. Moreover, in the discrete case we suppose that the matrix $(I - A_k)$ is nonsingular and its inverse we denote by $\tilde{A}_k$.

Linear Hamiltonian systems cover a large variety of linear equations. For example, the even order, self-adjoint, differential equation

$$\sum_{\nu=0}^{n} (-1)^\nu (r_\nu(t) y^{(\nu)})^{(\nu)} = 0 \tag{1}$$

may be written as LHS (H_c) using the substitution

$$x=\begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}, \quad u=\begin{pmatrix} (-1)^{n-1}(r_n y^{(n)})^{(n-1)}+\ldots+r_1 y' \\ \vdots \\ -(r_n y^{(n)})'+r_{n-1}y^{(n-1)} \\ r_n y^{(n)} \end{pmatrix}.$$

Then (x,u) solves (H_c) with A,B,C given by

$$B(t)=\mathrm{diag}\{0,\ldots,0,r_n^{-1}(t)\}, \quad C(t)=\mathrm{diag}\{r_0(t),\ldots,r_{n-1}(t)\},$$
$$A=A_{ij}=\begin{cases} 1, & \text{if } j=i+1,\ i=1,\ldots,n-1, \\ 0, & \text{elsewhere.} \end{cases}$$

Another example is the second order system

$$[R(t)x'+Q(t)x]'-\left[Q^T(t)x'+P(t)x\right]=0 \tag{2}$$

with $n\times n$ matrices P,Q,R, whereby P,R are symmetric and R nonsingular. Putting $u=R(t)x'+Q(t)x$, the pair of n-vectors (x,u) solves (H_c) with $B=R^{-1}$, $A=-R^{-1}Q$, $C=P-Q^TR^{-1}Q$. Discrete analogies of (1) and (2) can be written in the form (H_d) using essentially the same substitutions as in the continuous case.

Investigation of oscillatory properties of (H_c) has a relatively long history and was initiated by the paper of Morse [12] from 1930. Since that time oscillation theory of (H_c) attracted a considerable attention and the results of this investigation up to seventies can be found in the monograph of Reid [13]. In 1995 Kratz [11] published another comprehensive monograph which in addition to the classical results contains also the results achieved in the period 1980–95.

In contrast to the continuous case, oscillation theory of discrete systems (H_d) is much less developed and the fundamental result of this theory, a discrete version of the so-called Roundabout Theorem, was established only very recently by Bohner [5]. This paper accomplished the effort of several mathematicians in the last decade to prove the discrete Roundabout Theorem in its full generality, see [3].

Here we concentrate our attention to two problems of oscillation theory both of (H_c) and (H_d); to the so-called reciprocity principle and to trigonometric transformation. The paper is organized as follows. In the next section we recall basic facts of oscillation theory of continuous systems (H_c) and we formulate continuous versions of trigonometric transformation and reciprocity principle. In the last section we point out what difficulties one meets when trying to "discretize" these results, i. e., to extend them to (H_d).

2 Continuous Hamiltonian systems

We start with basic concepts of oscillation theory of (H_c).

Definition 1. Two points t_1, t_2 are said to be *conjugate* relative to (H_c) if there exists a solution (x, u) such that $x(t_1) = 0 = x(t_2)$ and $x(t) \not\equiv 0$ in $[t_1, t_2]$. System (H_c) is said to be *conjugate* in an interval $[a, b]$ if there exist $t_1, t_2 \in [a, b]$ which are conjugate relative to (H_c), in the opposite case (H_c) is said to be *disconjugate*. System (H_c) is said to be *oscillatory* if for every $c \in \mathbb{R}$ this system is conjugate in $[c, \infty)$, in the opposite case (H_c) is said to be *nonoscillatory*.

As mentioned in the previous section, principal statement concerning oscillatory properties of (H_c) is the so-called Reid Roundabout Theorem [13]. Before formulating it, we recall some very elementary properties of solutions of Hamiltonian systems.

Simultaneously with (H_c) we consider its matrix analogy

$$X' = A(t)X + B(t)U, \quad U' = C(t)X - A^T(t)U, \tag{3}$$

where X, U are $n \times n$ matrices. If $(X, U), (\tilde{X}, \tilde{U})$ are two solutions of (3) then the "Wronskian-type" identity $X^T\tilde{U} - U^T\tilde{X} \equiv K$ holds, where K is a constant $n \times n$ matrix. A solution (X, U) of (3) is said to be *conjoined* if X^TU is symmetric and it is said to be *conjoined basis* if, moreover, rank $(X^T, U^T) = n$.

Proposition 2. *(Reid [13]). The following statements are equivalent:*

(i) System (H_c) is disconjugate in the interval $[a, b]$.
(ii) The quadratic functional

$$\mathcal{F}(x, u) = \int_a^b [u^T(t)B(t)u(t) + x^T(t)C(t)x(t)]dt$$

is positive for every nontrivial (x, u) *satisfying* $x' = A(t)x + B(t)u$ *and* $x(a) = 0 = x(b)$.
(iii) The solution (X, U) *of (3) given by the initial condition* $X(a) = 0, U(a) = I$ *satisfies* $\det X(t) \neq 0$, $t \in (a, b]$.
(iv) There exists a conjoined basis (X, U) *of (3) such that* $X(t)$ *is nonsingular for* $t \in [a, b]$.
(v) There exists a symmetric matrix Q *which for* $t \in [a, b]$ *solves the Riccati matrix differential equation*

$$Q' = -C(t) + A^T(t)Q + QA(t) + QB(t)Q$$

related to (H_c) by the substitution $Q = UX^{-1}$.

For a better understanding of this statement we suggest the reader to see (H_c) as a rewritten second order equation (compare (1) with $n = 1$) and to compare this statement with the well known results of oscillation theory of second order equations.

Now we state some results concerning transformations of LHS. These results we need for formulation both of trigonometric transformation and reciprocity principle. A $2n \times 2n$ matrix $\mathcal{R}$ is said to be *symplectic* if $\mathcal{R}^T\mathcal{J}\mathcal{R} = \mathcal{J}$, where

$\mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, I being the $n \times n$ identity matrix. If $\mathcal{R} = \begin{pmatrix} H & K \\ M & N \end{pmatrix}$, where H, K, M, N are $n \times n$ matrices then $\mathcal{R}$ is symplectic if and only if

$$H^T K = K^T H, \; M^T N = N^T M, \; H^T N - K^T M = I. \tag{4}$$

Consider the transformation

$$\begin{pmatrix} x \\ u \end{pmatrix} = \mathcal{R}(t) \begin{pmatrix} y \\ z \end{pmatrix}, \text{ where } \quad \mathcal{R}(t) = \begin{pmatrix} H(t) & M(t) \\ K(t) & N(t) \end{pmatrix} \tag{5}$$

is symplectic and continuously differentiable. Then the variables y, z satisfy the LHS

$$(\bar{H}_c) \qquad y' = \bar{A}(t)y + \bar{B}(t)z, \quad z' = \bar{C}(t)y - \bar{A}^T(t)z,$$

where

$$\begin{aligned} \bar{A} &= N^T[-H' + AH + BK] - M^T[-K' + CH - A^T K], \\ \bar{B} &= N^T[-M' + AM + BN] + M^T[-N' + CM - A^T N], \\ \bar{C} &= -K^T[-H' + AH + BK] + H^T[-N' + CM - A^T N]. \end{aligned}$$

Observe that in the case $M(t) \equiv 0$ transformation (5) preserves oscillatory properties of transformed systems since then $H(t)$ is nonsingular (compare (4)), hence t_1, t_2 are conjugate relative to (H_c) if and only if they are conjugate relative to $(\bar{H}_c)$. Consequently, transformation (5) with $M(t) \equiv 0$ is the powerful tool for investigation of oscillatory properties of (H_c). This system is transformed into an "easier" system and from oscillatory properties of this "easy" system we deduce oscillatory properties of (H_c). One of such "easy systems" is the so-called trigonometric system.

Theorem 3. *(Došlý [7]). There exist continuously differentiable $n \times n$ matrices H, K such that H is nonsingular, $H^T K \equiv K^T H$ and the transformation*

$$\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} H(t) & 0 \\ K(t) & (H^T(t))^{-1} \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

transforms (H_c) into the trigonometric system

$$y' = Q(t)z, \quad z' = -Q(t)y, \tag{6}$$

where Q is a nonnegative definite symmetric $n \times n$ matrix.

The trigonometric system was introduced in [4] in connection with the Prüfer-type transformation for (H_c) and the terminology *trigonometric system* is justified by the fact that in the scalar case, i. e., when y, z, q are scalar quantities, then

$$(y_1, z_1) = (\sin \int^t q(s)\, ds, \cos \int^t q(s)\, ds), \quad (y_2, z_2) = (\cos \int^t q(s)\, ds, - \sin \int^t q(s)\, ds)$$

form the basis of the solution space of (6). In the n-dimensional case system (6) cannot be in general solved explicitly, but it may be proved that its solutions have many properties which in the scalar case reduce to the well-known trigonometric identities and these properties we may use to study properties of (H_c).

Now turn our attention to the reciprocity principle for LHS. We start with the following elementary example. Consider the second order equation

$$(r(t)y')' + p(t)y = 0 \tag{7}$$

with *positive* coefficients r, p. If we denote $z = r(t)y'$ then this function verifies the so-called reciprocal equation

$$\left(\frac{1}{p(t)}z'\right)' + \frac{1}{r(t)}z = 0. \tag{8}$$

Using an elementary argument it is easy to see that a solution y of (7) oscillates if and only if its derivatives y' oscillates, i. e., (7) is oscillatory if and only if (8) is oscillatory.

Reciprocity principle concerns extension of this statement to (H_c).

Theorem 4. *(Ahlbrandt [1]). Suppose that $B(t) \geq 0$, $C(t) \leq 0$ (this means that B is nonnegative definite and C nonpositive definite) for large t and both system (H_c) and its reciprocal system*

$$(\mathrm{R}_c) \qquad y' = -A^T(t)y - C(t)z, \quad z' = -B(t)y + A(t)z$$

are eventually controllable (i.e., the trivial solution $(x, u) = (0, 0)$ is the only solution of (H_c) for which one of the components x, u is eventually vanishing). Then (H_c) is oscillatory if and only if (R_c) is oscillatory.

Obviously, this statement is a generalization of the relationship between (7) and (8) and claims, roughly speaking, that (H_c) is oscillatory with respect to the first component x if and only if it is oscillatory with respect to the second component u (compare Definition 1). Indeed, (R_c) results from (H_c) upon the transformation

$$\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \mathcal{J} \begin{pmatrix} y \\ z \end{pmatrix} \tag{9}$$

which essentially only reverses the order of equations in (H_c).

3 Discrete Hamiltonian systems

Similar to the continuous case, we start with the definition of basic concepts.

Definition 5. We say that the interval $(k, k+1]$, $k \in \mathbb{N}$, contains a *generalized zero* of a solution (x, u) of (H_d) if $x_k \neq 0$, there exists $c \in \mathbb{R}^n$ such that

$$x_{k+1} = \tilde{A}_k B_k c, \quad \text{and} \quad c^T B_k^{\dagger}(I - A_k)x_k \leq 0.$$

System (H_d) is said to be *disconjugate* in an interval $[n,m]$ if any solution of (H_d) has at most one generalized zero in $[n,m+1]$ and, moreover, any solution satisfying $x_n=0$ has no generalized zero in $(n,m+1]$, in the opposite case (H_d) is said to be *conjugate* in $[n,m]$. System (H_d) is said to be *nonoscillatory* if there exists $n\in\mathbb{N}$ such that (H_d) is disconjugate on $[n,m]$ for every $m>n$, in the opposite case (H_d) is said to be *oscillatory.*

In the above definition † denotes the Moore-Penrose generalized inverse matrix, for an $n\times n$ matrix V its generalized inverse $V^\dagger$ is the (unique) $n\times n$ matrix such that $VV^\dagger$, $V^\dagger V$ are symmetric and $V^\dagger V V^\dagger = V^\dagger$, $VV^\dagger V=V$.

Basic oscillatory properties of discrete Hamiltonian systems are summarized in the discrete version of Roundabout Theorem.

Proposition 6. *(Bohner [6]). The following statements are equivalent:*

(i) System (H_d) is disconjugate in the interval $[0,N]$, $N\in\mathbb{N}$.

(ii) The discrete quadratic functional

$$\mathcal{F}(x,u)=\sum_{k=0}^{N}\left\{u_k^T B_k u_k + x_{k+1}^T C_k x_{k+1}\right\}$$

is positive for every nontrivial (x,u) *satisfying* $\Delta x_k = A_k x_{k+1}+B_k u_k$ *and* $x_0=0=x_{N+1}$.

(iii) The matrix solution (X,U) *of (H_d) given by the initial condition* $X_0=0$, $U_0=I$ *satisfies*

$$\operatorname{Ker} X_{k+1}\subseteq \operatorname{Ker} X_k \quad\text{and}\quad X_k X_{k+1}^\dagger \tilde{A}_k B_k\ge 0,\ k=1,\dots,N.$$

(iv) There exists a conjoined basis (X,U) *of (H_d) (this is defined in the same way as for (H_c)) such that* X_k *are nonsingular and* $X_k X_{k+1}^{-1}\tilde{A}_k B_k\ge 0,\ k=0,\dots,N.$

(v) There exist symmetric matrices Q_k *such that* $(I+B_kQ_k)$ *are nonsingular,* $(I+B_kQ_k)^{-1}B_k\ge 0$, *and verify the discrete Riccati matrix difference equation*

$$Q_{k+1}=C_k+(I-A_k^T)Q_k(I+B_kQ_k)^{-1}(I-A_k),$$

$k=0,\dots,N$.

Concerning transformations of discrete LHS (H_d), the situation is not so easy as in the continuous case. In the discrete case we must suppose that the matrices $(I-A_k)$ are nonsingular and this assumption must satisfy also the system after transformation. To ensure this, we need an extra assumption as shows the next theorem.

Theorem 7. *(Došlý [8]). Let* $\mathcal{R}_k$ *be a* $2n\times 2n$ *symplectic matrix consisting of* $n\times n$ *matrices* $\mathcal{R}_k=\begin{pmatrix}H_k & M_k\\ K_k & N_k\end{pmatrix}$ *such that the matrix*

$$\begin{pmatrix}H_k+B_kK_k & (I-A_k)M_{k+1}\\ (I-A_k^T)K_k & N_{k+1}-C_kM_{k+1}\end{pmatrix} \tag{10}$$

is nonsingular and denote $\begin{pmatrix} D_k & F_k \\ E_k & G_k \end{pmatrix}$ *its inverse. The transformation*

$$\begin{pmatrix} x \\ u \end{pmatrix} = \mathcal{R}_k \begin{pmatrix} y \\ z \end{pmatrix} \tag{11}$$

transforms (H_d) *into the system*

$$(\bar{\mathrm{H}}_d) \qquad \Delta y_k = \bar{A}_k y_{k+1} + \bar{B}_k z_k, \quad \Delta z_k = \bar{C}_k y_{k+1} - \bar{A}_k^T z_k,$$

where

$$\begin{aligned}
\bar{A}_k &= D_k(-\Delta H_k + A_k H_{k+1} + B_k K_k) + F_k(-\Delta K_k + C_k H_{k+1} - A_k^T K_k), \\
\bar{B}_k &= D_k(-\Delta M_k + A_k M_{k+1} + B_k N_k) + F_k(-\Delta N_k + C_k M_{k+1} - A_k^T N_k), \\
\bar{C}_k &= E_k(-\Delta H_k + A_k H_{k+1} + B_k K_k) + G_k(-\Delta K_k + C_k H_{k+1} - A_k^T K_k),
\end{aligned}$$

in particular, the matrices $\bar{B}_k, \bar{C}_k$ *are symmetric and* $(I - \bar{A}_k)$ *are nonsingular, i.e.,* ($\bar{H}_d$) *is again a difference LHS.*

Having now in disposal the above given statements, we may try to extend reciprocity principle and trigonometric transformation to discrete systems. Let us start with reciprocity principle. If we apply transformation (11) with $\mathcal{R} = \mathcal{J}$ (this transformation in the continuous case relates (H_c) and (R_c)) to (H_d), it is easy to see that the assumption of Theorem 7 concerning nonsingularity of the matrix in (10) is not generally satisfied, i.e. the resulting (reciprocal) system

$$(\mathrm{R}_d) \qquad \Delta y_k = -A_k^T y_k - C_k z_{k+1}, \quad \Delta z_k = -B_k y_k + A_k z_{k+1}$$

is the system of a different kind than (H_d). In fact, the variable x which defines oscillatory properties of (H_d) appears in the right-hand-side of this system with indices $k+1$, whereas the variable y which should define oscillations of (R_d) appears there with indices k. For this reason, Definition 2 and Proposition 6 do not apply to (R_d). However, as suggests the equivalence between oscillatory properties of the pair of second order equations $\Delta(r_k \Delta x_k) + p_k x_{k+1} = 0$ and $\Delta(p_k^{-1}\Delta z_k) + r_{k+1}^{-1} z_{k+1} = 0$, which follows using the same argumentation as in the continuous case, we may expect some kind of similarity between oscillatory properties of (H_d) and (R_d).

In studying the relationship between (H_d) and (R_d), the principal role play the so-called *symplectic systems*, i. e. systems of the form

$$(S) \qquad \begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = S_k \begin{pmatrix} x_k \\ u_k \end{pmatrix}, \quad S_k = \begin{pmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{pmatrix},$$

where S_k are symplectic $2n \times 2n$ matrices. Expanding forward differences in (H_d) and (R_d), it is not difficult to see that these systems are symplectic systems. Oscillation theory of symplectic systems was established in [7] and fundamental definition is the following:

Definition 8. We say that the interval $(k,k+1]$ contains the *generalized zero* of a solution (x,u) of (S) if $x_k \neq 0$, there exists $c \in \mathbb{R}^n$ such that

$$x_{k+1} = \mathcal{B}_k c \quad \text{and} \quad c^T \mathcal{B}_k^{\dagger} x_k \leq 0.$$

Oscillation and nonoscillation of symplectic systems are defined via generalized zeros in the same way as for Hamiltonian systems. Applying these definitions to (H_d) and (R_d) we get the following discrete version of the reciprocity principle.

Theorem 9. *(Došlý-Bohner [6]). Suppose that $B_k \geq 0$, $C_k \leq 0$, for large k, and both (H_d) and (R_d) are eventually controllable. Then (H_d) is oscillatory if and only if (R_d) is oscillatory.*

Now let us pass to trigonometric transformation. In the continuous case this transformation may be regarded as a transformation which transforms (H_c) into a system which complies with its reciprocal system. This observation shows that to find a discrete analogy of this statement will not be easy. As "the first approximation" of the problem consider the pair of second order equations

$$(\mathrm{E}_{c,d}) \qquad x'' + p(t)x = 0 \quad \text{and} \quad \Delta^2 x_k = p_k x_{k+1} = 0.$$

Let x_1, x_2 be solutions of (E_c) and consider the transformation $x = \sqrt{x_1^2 + x_2^2}\, y$. This is just the trigonometric transformation of (E_c) which transforms this equation into the equation

$$(q^{-1}(t)y')' + q(t)y = 0$$

with $q = (x_1^2 + x_2^2)^{-2}$, its solutions are $y_1 = \sin \int_{t_0}^t q(s)\,ds$, $y_2 = \cos \int_{t_0}^t q(s)\,ds$. The function $\alpha(t) = \int_{t_0}^t (x_1^2 + x_2^2)^{-2}\,ds$ is sometimes called the *zero-counting* function since each its crossing of a multiple of π gives a zero point of any solution of (E_c) satisfying $y(t_0) = 0$.

Let y_k, z_k be solutions of (E_d) and let $h_k = \sqrt{y_k^2 + z_k^2}$. If we try to modify the method used for (E_c) to discrete equation (E_d), after some computation we find that the only "discrete candidate" which could play the role of the function $x_1^2 + x_2^2$, is the sequence $y_k y_{k+1} + z_k z_{k+1}$. However, in contrast to the continuous case where the function $x_1^2 + x_2^2$ is positive and hence α is increasing, the sequence $y_k y_{k+1} + z_k z_{k+1}$ may change its sign, may even be identically zero as shows the simple equation $\Delta^2 x_k + 2x_{k+1} = 0$. This leads to the following conclusion:

No discrete analogy of the trigonometric transformation is known even for the simple second order equation (E_d).

Our investigation of relations between oscillation theories of discrete and continuous systems we may conclude with the following brief summary. In some problems, mostly based on equivalence of (i) and (ii) in the Roundabout Theorem, one can find full analogy between discrete and continuous statements. On the other hand, in some problems one needs to establish completely new methods (oscillation theory of symplectic systems in extension of the reciprocity principle) and, in some cases, till now no discrete analogy of "continuous" statements is known as in the case of trigonometric transformation.

Supported by the Grant No. 201/96/0410 of the Czech Grant Agency (Prague)

References

1. C. D. Ahlbrandt, Equivalent boundary value problems for self-adjoint differential system, *J. Diff. Equations*, **9** (1971), 420–435.
2. C. D. Ahlbrandt, Recessive solutions of symmetric three term recurrence relations, *Canad. Math. Soc. Proceedings*, **8** (1987), 3–42.
3. C. D. Ahlbrandt, A. C. Peterson, *Discrete Hamiltonian Systems: Difference Equations, Continued Fractions, and Riccati Equations*, Kluwer, Boston 1996.
4. J. H. Barrett, A Prüfer transformation for matrix differential system, *Proc. Amer. Math. Soc.*, **8** (1957), 510–518.
5. M. Bohner, Linear Hamiltonian difference systems: disconjugacy and Jacobi-type conditions, *J. Math. Anal. Appl.*, **199** (1996), 804–826.
6. M. Bohner, O. Došlý, Disconjugacy and transformations for symplectic systems, to appear in *Rocky Mountain J. Math.*
7. O. Došlý, On transformations of self-adjoint linear differential systems and their reciprocals, *Annal. Pol. Math.*, **50** (1990), 223–224.
8. O. Došlý, Transformations of linear Hamiltonian difference systems and some of their applications, *J. Math. Anal. Appl.*, **191** (1995), 250–265.
9. L. Erbe, P. Yan, Disconjugacy for linear Hamiltonian difference systems, *J. Math. Anal. Appl.*, **167**, (1992), 355–367.
10. L. Erbe, P. Yan, Qualitative properties of Hamiltonian difference systems, *J. Math. Anal. Appl.*, **171**, (1992), 334–345.
11. W. Kratz, *Quadratic Functionals in Variational Analysis and Control Theory*, Akademie Verlag, Berlin, 1995
12. M. Morse, A generalization of Sturm separation and comparison theorems in n-space, *Math. Ann.*, **103** (1930), 52–69.
13. W. T. Reid, *Sturmian Theory for Ordinary Differential Equations*, Springer Verlag, New York-Heidelberg-Berlin 1980.