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Approximating Weak Bisimulation on Basic Process Algebra

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Abstract

The maximal strong and weak bisimulations on any class of processes can be obtained as the limits of decreasing chains of binary relations, *approximants*. In the case of strong bisimulation and Basic Process Algebras this chain has length at most ω which enables semidecidability of strong bisimilarity. We show that it is not so for weak bisimulation where the chain can grow much longer, and discuss the implications this has for the problem of (semi)decidability of weak bisimilarity.

1 Introduction

Algebraic descriptions are often used in concurrency theory for specification and verification of concurrent systems. There exist powerful process calculi such as CCS [7] that are very expressive. However, these calculi have the disadvantage that testing a property of a system may become infeasible. Simpler classes of processes are often built around fewer operators and do not possess the full expressive power but they enable more efficient testing. One class of processes that has been studied a lot recently describes systems that can compose in a sequential manner. These are Basic Process Algebras (BPA), originally introduced in [1] as the process equivalent of context-free grammars. Although the structure of BPA-processes is simple they can be used to describe quite a large class of infinite behaviours.

One of the properties of systems that we are interested in is *behavioural equivalence*. For the sake of system design and verification we need to be able to specify and test when processes are equivalent with respect to some notion of observation. Among the most favoured behavioural equivalences are *strong* and *weak bisimulations*, introduced in [7]. One of the main issues for equivalences is the decidability problem: we want to determine whether a particular equivalence can be decided for any pair of processes from a fixed class.

Strong bisimulation equivalence can be decided on the class of BPA-processes (see e.g. [2], [3]). The classical test ([3]) consists of two semidecision procedures. The algorithm for semideciding strong bisimilarity is based on the fact that for every BPA there exists a finite *base* from which all bisimilar pairs can be derived. The algorithm for semideciding non-bisimilarity approximates the maximal bisimulation equivalence with a possibly infinite decreasing sequence of binary equivalences that always converge, with the limit being the maximal bisimulation equivalence.

Whereas the notion of strong bisimilarity is based on actions that systems perform, weak bisimilarity takes into account only actions of a system that are *observable*. Processes can thus engage in *internal* transitions that cannot be seen by an outside observer, and hence do not have to be matched by an equivalent process. A special *silent* action τ is set apart to denote internal behaviour and any *external* (observable) transition can be preceded and followed with an arbitrary sequence of $\xrightarrow{\tau}$. The drawback of abstracting away from internal transitions is that even simple systems as BPA may become infinitely branching, i.e. the transition (derivation) trees determined by BPA-processes may contain infinite branching. This poses a potential complication to equivalence testing. Indeed, even for the rather simple processes of BPA the decidability problem for weak bisimulation equivalence remains open.

At present there are some partial results that assert decidability of weak bisimilarity on strict subclasses of BPA-processes ([5], [9]). However, these are subclasses where the power of internal behaviour is in some ways restricted. In the general case neither weak bisimilarity nor weak non-bisimilarity can be semidecided. In this paper we concentrate on the technique for semideciding non-equivalence that is used for strong bisimilarity and we show that no straightforward application to weak bisimilarity seems possible.

2 Background

In order to define Basic Process Algebras we presuppose a fixed set of *actions* $Act = \{a, b, c, \ldots\}$ that contains a special action τ, and a finite set of process variables or atoms $\Sigma = \{X_1, \ldots, X_n\}$. A *Basic Process Algebra (BPA)* is then a pair (Σ^*, Δ), where Σ^* is the free monoid generated by Σ, and $\Delta = \{X \xrightarrow{\mu} P \mid X \in \Sigma, P \in \Sigma^*, \mu \in Act\}$ is a finite set of transitions. BPA-processes are identified with words from Σ^*. The transition rules of Δ determine a transition relation on general BPA-processes in this way:

$$XQ \xrightarrow{\mu} PQ \text{ if there is a rule } X \xrightarrow{\mu} P \text{ in } \Delta.$$

We will use capital letters X, Y to range over process variables, P, Q, R to range over BPA-processes, and μ, λ to range over actions.

Example 1 *Here we present a simple BPA. The set of variables Σ is $\{X, Y\}$ and the transition rules of Δ are given below:*

$$Y \xrightarrow{a} \epsilon \qquad X \xrightarrow{\tau} XY \qquad X \xrightarrow{b} \epsilon.$$

The transition tree determined by the variable X is sketched in figure 1.

$$\begin{tikzcd}
X \arrow[r, "\tau"] \arrow[d, "b"'] & XY \arrow[r, "\tau"] \arrow[d, "b"'] & XY^2 \arrow[rr, dotted] \arrow[d, "b"'] & & XY^n \arrow[r, "\tau"] \arrow[d, "b"'] & XY^{n+1} \arrow[r, dotted] \arrow[d, "b"'] & {} \\
\epsilon & Y \arrow[d, "a"'] & Y^2 \arrow[d, "a"'] & \cdots & Y^n \arrow[d, "a"'] & Y^{n+1} \arrow[d, "a"'] & \cdots \\
& \epsilon & Y \arrow[d, "a"'] & \cdots & Y^{n-1} \arrow[d, dotted] & Y^n \arrow[d, dotted] & \cdots \\
& & \epsilon & & {} & {} &
\end{tikzcd}$$

Figure 1: The transition tree of the process X

The process X can with a sequence of n transitions $\xrightarrow{\tau}$ generate n copies of Y thus becoming XY^n. For XY^n to perform any move of Y the process has to dispose of the X in front with an $X \xrightarrow{b} \epsilon$ move. Only then can an a action of Y be done *as it is always the leftmost variable in a BPA-process that is allowed to carry out a transition.* $\square$

In order to incorporate the notion of internal behaviour we consider composite actions $\stackrel{\mu}{\Longrightarrow}$, where $\stackrel{\mu}{\Longrightarrow}$ is an abbreviation of $(\stackrel{\tau}{\longrightarrow})^* \stackrel{\mu}{\longrightarrow} (\stackrel{\tau}{\longrightarrow})^*$ in case $\mu \neq \tau$, and $(\stackrel{\tau}{\longrightarrow})^*$ in case $\mu = \tau$. The process X from Example 1 gives rise to an infinitely branching tree that is shown in figure 2.

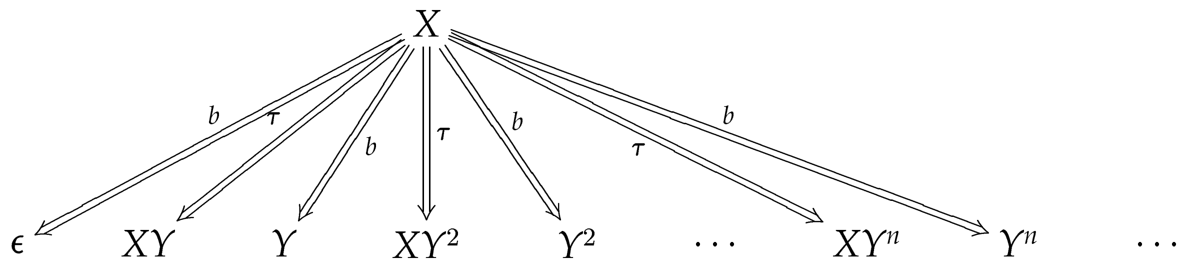

Figure 2: X as an infinitely branching tree

We say that a binary relation $\mathcal{R}$ on processes is a *weak bisimulation* if for every pair (P, Q) from $\mathcal{R}$ and every action μ from *Act* the following holds:

- for every $P \stackrel{\mu}{\Longrightarrow} P'$ there exists $Q \stackrel{\mu}{\Longrightarrow} Q'$ so that $(P', Q') \in \mathcal{R}$;
- for every $Q \stackrel{\mu}{\Longrightarrow} Q'$ there exists $P \stackrel{\mu}{\Longrightarrow} P'$ so that $(P', Q') \in \mathcal{R}$.

Two processes P and Q are *weakly bisimilar* if there exists a weak bisimulation containing the pair (P, Q). The union of all weak bisimulations gives rise to the maximal weak bisimulation which is denoted by $\approx$. An equivalent definition of weak bisimulation is phrased in terms of single transition in the premise followed by composite transition. Both definitions yield identical maximal weak bisimulations and we shall be using either of them, depending on the context.

In the definition above, the maximal weak bisimulation is obtained as the union of smaller weak bisimulations. There exists an alternative approach (see Milner [7]) where the maximal equivalence is obtained as the limit of a decreasing chain of *weak bisimulation approximants*. These are binary relations on processes defined inductively in terms of *ordinal numbers*.

For the sake of simplicity, in this paper we shall view ordinals as generalisation of (the well-ordered nature of) natural numbers. Ordinal numbers form a class, denoted by *On*, and are well-ordered by the element-of relation $<$. The initial segment of *On* containing natural numbers and ω is $0, 1, 2, \ldots, n, \ldots, \omega$, after which follow $\omega+1, \omega+2, \ldots, \omega+n, \ldots, \omega+\omega$, etc. The ordinals $0, \omega, \omega+\omega$, are *limit* ordinals which means they have no predecessor, whereas ordinals such as $1, 2$, and $\omega+1, \omega+2$, are *successor* ordinals. We shall be using some simple arithmetical operations on ordinals such as summation and multiplication, as well as the principle of *transfinite induction,* the induction principle generalised to the class of all ordinal numbers. Ordinals shall also provide us with a measure for derivation trees of particular processes which will implicitly refer to the notion of *rank (height)* of a tree. For a detailed instruction on ordinal numbers the reader should consult standard textbooks on set theory, such as [6].

Weak bisimulation approximants are defined inductively on the class *On* in this way:

- $P \approx_0 Q$ for all P and Q;
- $P \approx_{\alpha+1} Q$ if for all actions μ,
  - whenever $P \stackrel{\mu}{\Longrightarrow} P'$ then there exists $Q \stackrel{\mu}{\Longrightarrow} Q'$ so that $P' \approx_\alpha Q'$;
  - whenever $Q \stackrel{\mu}{\Longrightarrow} Q'$ then there exists $P \stackrel{\mu}{\Longrightarrow} P'$ so that $P' \approx_\alpha Q'$;
- $P \approx_\lambda Q$ if $P \approx_\alpha Q$ for every $\alpha < \lambda$, for a limit ordinal λ.

It can be easily verified that binary relations $\approx_\alpha$ are equivalences for every ordinal α. The following lemma sums up the structure of the chain of approximants and the relationship between individual approximants and the maximal bisimulation. A proof can be found in [7], [10].

Lemma 2

1. *for every* $\alpha, \beta \in On$, $\alpha < \beta \implies \approx_\beta \subseteq \approx_\alpha$;
2. *for every* $\alpha \in On$, $\approx \subseteq \approx_\alpha$;
3. *if there is an* α *such that* $\approx_\alpha = \approx_{\alpha+1}$ *then for all* $\beta \geq \alpha$, $\approx_\alpha = \approx_\beta = \approx$;
4. $\approx = \bigcap_{\alpha \in On} \approx_\alpha$.

1 says that approximants form a non-increasing sequence. 2 says that the maximal equivalence is contained in every approximant. 3 and 4 state that the sequence converges and the limit is $\approx$.

Note: The notion of strong bisimulation $\sim$, resp. strong bisimulation approximants $\sim_\alpha$, is defined analogously to weak bisimulation, resp. weak bisimulation approximants, where the composite transition $\stackrel{\mu}{\Longrightarrow}$ is replaced by the single transition $\stackrel{\mu}{\longrightarrow}$. A lemma analogous to Lemma 2 holds, i.e. the chain of strong bisimulation approximants converges with the limit being $\sim$. For finitely branching processes (such as BPA-processes) the convergence occurs at level ω, that is $\sim\, = \,\sim_\omega\, = \,\bigcap_{n\in\omega} \sim_n$. Proof of this claim can be found in [4].

Every BPA-process has only finitely many possible derivatives therefore each approximant $\sim_n$ is decidable. Then a straightforward semidecision procedure for non-bisimilarity proceeds by successive enumeration of all natural numbers n and testing equivalence at $\sim_n$. However, we shall see that this approach cannot be used for weak bisimulation approximants as we shall establish that there exist BPA where $\approx\, \subsetneq\, \approx_\omega$.

3 Hierarchy of Non-bisimilar BPA-processes

In this section we will construct a hierarchy of processes that will distinguish individual approximants $\approx_{\omega^n}$ (and all approximants in between) from the maximal weak bisimulation $\approx$. We will start with a simple construction that will later lend itself to straightforward generalisation. We define variables C and A by these transition rules:

$$A \stackrel{a}{\longrightarrow} \epsilon \quad A \stackrel{\tau}{\longrightarrow} \epsilon \quad C \stackrel{\tau}{\longrightarrow} CA \quad C \stackrel{\tau}{\longrightarrow} \epsilon.$$

The transition tree determined by the variable C is drawn in figure 3. The tree contains infinite branching at the top level as C can generate with a $\stackrel{\tau}{\Longrightarrow}$ move any number of copies of A. It is not difficult to see that $A^k \approx_k A^l$ for every $k \leq l$. In order to distinguish $\approx_\omega$ from $\approx$ it suffices to consider the processes C and AC.

Lemma 3 *$C \approx_\omega AC$ and $C \not\approx AC$.*

Proof: We will show that the processes C and AC are equivalent at $\approx_k$ for all k but not weakly bisimilar. Any move of C can be matched by AC after discarding the A in front with $AC \xrightarrow{\tau} C$. To the move $AC \xrightarrow{\tau} C$ the variable C responds with $C \overset{\epsilon}{\Longrightarrow} C$. Hence we only need to consider the $\xrightarrow{a}$ transition of AC. For a fixed k, if $AC \xrightarrow{a} C$ then C generates enough copies of A with $C \xrightarrow{\tau} A^{N+1} \xrightarrow{a} A^N$, for some $N > k$. Then C and A^N will surely be related at $\approx_k$ and we can conclude that $C \approx_\omega AC$. On the other hand, because AC has one more copy of A at its disposal, we obtain that $C \not\approx_{\omega+1} AC$ and hence $C \not\approx AC$. $\square$

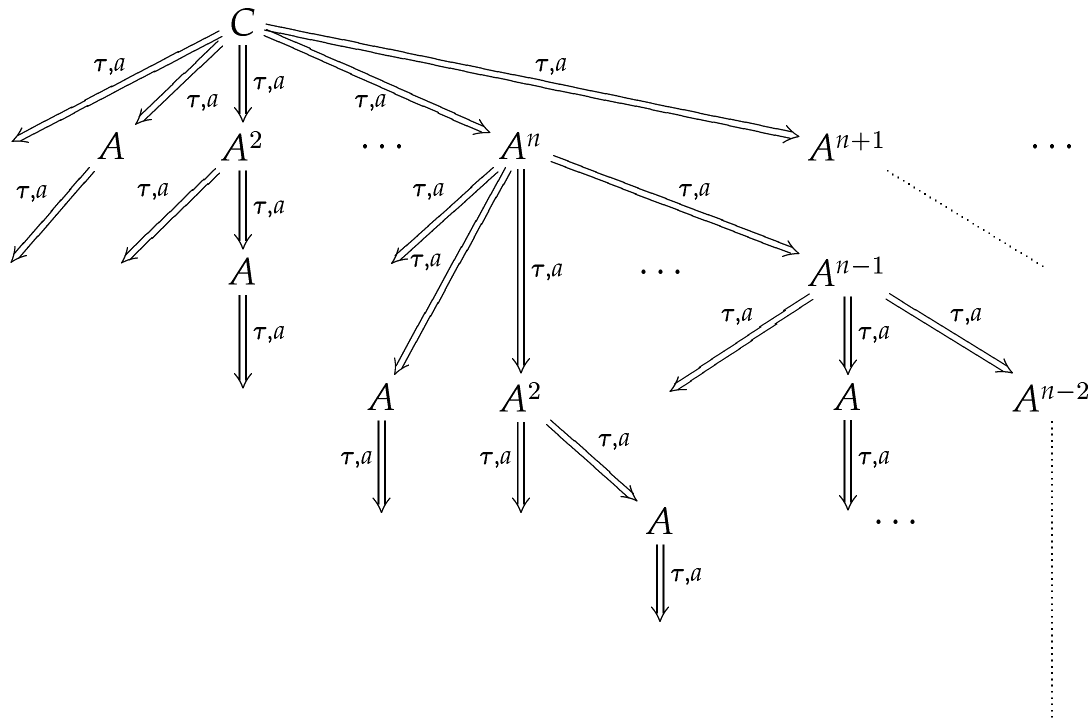

Figure 3: The process C

We can use the two variables C and A to reach even higher. If we consider the processes C^n and AC^n for some $n > 1$, then we can repeat the trick of generating arbitrary many copies of A several (at most n) times. Hence each copy of C gives rise to an infinitely branching tier in the derivation tree determined by the process C^n, resp. AC^n. Thus we construct trees of height $\omega \cdot n$, resp. $\omega \cdot n + 1$. The process AC^n has one extra action compared with C^n which makes the two processes non-bisimilar while being equivalent at the level $\approx_{\omega \cdot n}$.

In the same way that C could generate any number of atoms A we can introduce a new variable capable of producing any number of C's. Thus we can obtain an infinite hierarchy of variables that will enable us to go beyond ω^2. The variables are defined inductively in this way:

1. $D_0 \xrightarrow{a} \epsilon$, $D_0 \xrightarrow{\tau} \epsilon$;

2. assuming we have defined the variables $D_0, \ldots, D_i$, the variable D_{i+1} is defined by $D_{i+1} \xrightarrow{\tau} D_{i+1}D_i$, $D_{i+1} \xrightarrow{\tau} \epsilon$.

The variable A, resp. C, is renamed D_0, resp. D_1. Notice that the only variable capable of performing a visible action is D_0. The purpose of the other variables is to create bigger and bigger branching so as to obtain trees of still greater height. The ultimate goal is to construct pairs of processes P_i and Q_i, sequences over variables $D_0, D_1, \ldots, D_i$, with the property that $P_i \approx_{\omega^i} Q_i$ and $P_i \not\approx Q_i$. Before we carry out the desired construction we shall analyse the behaviour of individual variables D_i.

Starting from a variable D_i we can only do a $\stackrel{\tau}{\Longrightarrow}$ sequence with which we obtain the process $D_i D_{i-1}^{e_{i-1}}$ for some e_{i-1}. We cannot get a more complex shape without removing the D_i in front. After having discarded D_i we can continue and from $D_{i-1}^{e_{i-1}}$ generate (with another $\stackrel{\tau}{\Longrightarrow}$ sequence) the process $D_{i-1}D_{i-2}^{e_{i-2}}D_{i-1}^{e_{i-1}-1}$. We repeat the procedure several times and finally derive a process either of the form $D_{k+1}D_k^{e_k}D_{k+1}^{e_{k+1}} \ldots D_m^{e_m}$ or $D_k^{e_k}D_{k+1}^{e_{k+1}} \ldots D_m^{e_m}$, where $k \geq 0, m < i$ and $e_k, \ldots, e_m \geq 0$. The latter process is a composition of atoms with increasing indeces and so will be denoted by $\prod_{i=0}^{m} D_i^{e_i}$ and called a *product*. We will see that it suffices to consider products as it is not difficult to convince oneself that every $D_{k+1}D_k^{e_k}D_{k+1}^{e_{k+1}} \ldots D_m^{e_m}$ is weakly bisimilar to the product $D_{k+1}^{e_{k+1}+1}D_{k+2}^{e_{k+2}} \ldots D_m^{e_m} \equiv D_{k+1}^{e_{k+1}+1} \prod_{i=k+2}^{m} D_i^{e_i}$. That means these two processes produce identical behaviour, with regard to $\approx$ and also $\approx_\alpha$. First we shall prove the following statement that characterises one particular type of weakly bisimilar processes over atoms D_i:

Proposition 4 *For every k, m, and l, $D_{k+1}^{l+1} \approx D_{k+1}D_k^m D_{k+1}^l$.*

Proof: In order to demonstrate that two processes are weakly bisimilar it suffices to construct a binary relation containing the pair of the processes in question and show that the relation is a weak bisimulation. For that purpose we will define a binary relation $\mathcal{R} = \{(D_l D_{l-1}^m D_l^k, D_l D_{l-1}^n D_l^k) | k, l > 0, m, n \in \mathbb{N}\} \cup \{(\hat{D}, \hat{D}) | D_l D_{l-1}^m D_l^k \stackrel{a^*}{\Longrightarrow} \hat{D}, k, l > 0, m \in \mathbb{N}\}$. Clearly the relation $\mathcal{R}$ contains the pairs $(D_l^{k+1}, D_l D_{l-1}^m D_l^k)$ for every k, m and $l > 0$. Now we have to check that it is closed under expansion with $\stackrel{\mu}{\longrightarrow}$, that means for every pair (P, Q) from $\mathcal{R}$, if there is a transition $P \stackrel{\mu}{\longrightarrow} P'$ then there has to be a matching transition $Q \stackrel{\mu}{\Longrightarrow} Q'$ with the resulting pair (P', Q') again in $\mathcal{R}$, and conversely, also starting from Q.

Firstly we will choose a pair $(D_l D_{l-1}^m D_l^k, D_l D_{l-1}^n D_l^k)$ for some fixed $k, l > 0, m$ and n. We remind ourselves that for any $l > 0$, the only possible transitions D_l can do are $D_l \stackrel{\tau}{\longrightarrow} \epsilon$ and $D_l \stackrel{\tau}{\longrightarrow} D_l D_{l-1}$. If either $D_l D_{l-1}^m D_l^k$ or $D_l D_{l-1}^n D_l^k$ chooses to perform the transition $D_l \stackrel{\tau}{\longrightarrow} D_l D_{l-1}$ the other process does exactly the same which results in a pair $(D_l D_{l-1}^{m+1} D_l^k, D_l D_{l-1}^{n+1} D_l^k)$ that belongs to $\mathcal{R}$ by definition.

To analyse the case when one process decides to employ the transition $D_l \stackrel{\tau}{\longrightarrow} \epsilon$ we will assume that $m \leq n$. Hence the response to the move $D_l D_{l-1}^n D_l^k \stackrel{\tau}{\longrightarrow} D_{l-1}^n D_l^k$ will be $D_l D_{l-1}^m D_l^k \stackrel{\tau^{n-m}}{\longrightarrow} D_l D_{l-1}^n D_l^k \stackrel{\tau}{\longrightarrow} D_{l-1}^n D_l^k$ and the pair $(D_{l-1}^n D_l^k, D_{l-1}^n D_l^k)$ will belong to $\mathcal{R}$ since $D_{l-1}^n D_l^k$ is derived from $D_l D_{l-1}^n D_l^k$. If it is $D_l D_{l-1}^m D_l^k$ that disposes of D_l and becomes $D_{l-1}^m D_l^k$ then the other process $D_l D_{l-1}^n D_l^k$ responds by removing D_l in the first place and then all superfluous copies of D_{l-1} to become $D_{l-1}^m D_l^k$. Again the pair $(D_{l-1}^m D_l^k, D_{l-1}^m D_l^k)$ is in $\mathcal{R}$.

Lastly, if we have a pair $(\hat{D}, \hat{D})$ from $\mathcal{R}$ with $\hat{D}$ being an $\stackrel{a^*}{\Longrightarrow}$ derivative of some $D_l D_{l-1}^m D_l^k$ then any $\bar{D}$ obtained from $\hat{D}$ by performing $\stackrel{\mu}{\longrightarrow}$ is also an $\stackrel{a^*}{\Longrightarrow}$ derivative of $D_l D_{l-1}^m D_l^k$ and hence the pair $(\bar{D}, \bar{D})$ belongs to $\mathcal{R}$. □

As a corollary of Proposition 4 we obtain the desired relation between the two types of processes that can arise as product derivatives.

Corollary 5 *The processes $D_l \prod_{i=l-1}^m D_i^{f_i}$ and $D_l^{f_l+1} \prod_{i=l+1}^m D_i^{f_i}$ are weakly bisimilar.*

In fact, with a bit extra effort one could show that an arbitrary composition of the atoms D_i is weakly bisimilar to some product, however we shall not pursue that line here.

Next we will define a measure on products that will enable us to make statements about the largest ordinal number that relates two non-bisimilar products. The measure is not chosen arbitrarily but captures in some way the branching power that processes possess; in fact it corresponds precisely to the ordinal height of derivation trees of product processes. To a product $\prod_{i=0}^{m} D_i^{e_i}$ we assign an ordinal number $\sum_{i=0}^{m} \omega^i e_i = \omega^m e_m + \omega^{m-1} e_{m-1} + \ldots + \omega e_1 + e_0$. As will be shown later, one special property of this notion is that all derivatives of a product are assigned a smaller or equal ordinal.

Example 6 *We consider the variables D_2, D_1 and D_0. Starting from D_2, we can perform this derivation sequence: $D_2 \xrightarrow{\tau}^3 D_2 D_1^3 \xrightarrow{\tau} D_1^3 \xrightarrow{\tau}^5 D_1 D_0^5 D_1^2 \xrightarrow{\tau} D_0^5 D_1^2 \xrightarrow{a} D_0^4 D_1^2 \ldots$. On the other hand, there is no derivation sequence that would produce the process $D_2 D_1 D_0$. The ordinals assigned to each element of the derivation sequence are ω^2 to D_2, then $\omega \cdot 3$ to D_1^3, $\omega \cdot 2 + 5$ to $D_0^5 D_1^2$, and $\omega \cdot 2 + 4$ to $D_0^4 D_1^2$. Finally, processes $D_1^3 D_0^5$ and $D_1 D_0^5 D_1^2$ are weakly bisimilar.* $\square$

Finally we can give a precise classification of product derivatives in terms of the ordinal measure defined above. We shall rely on the following proposition in the proofs of the next section where a rigourous analysis of product behaviour shall be required.

Proposition 7 *For a product $\prod_{i=0}^{m} D_i^{e_i}$ and process P, if there is a derivation $\prod_{i=0}^{m} D_i^{e_i} \stackrel{\mu}{\Longrightarrow} P$ then P is weakly bisimilar to some $\prod_{i=0}^{m} D_i^{f_i}$, where $\sum_{i=0}^{m} \omega^i f_i < \sum_{i=0}^{m} \omega^i e_i$ in case $\mu = a$, and $\sum_{i=0}^{m} \omega^i f_i \leq \sum_{i=0}^{m} \omega^i e_i$ in case $\mu = \tau$.*

Proof: We assume a fixed product $\prod_{i=0}^{m} D_i^{e_i}$ and the corresponding $\sum_{i=0}^{m} \omega^i e_i$. If $e_0 > 0$ then before removing all copies of D_0, all sequences of transitions lead to a product of the form $D_0^e D_1^{e_1} \ldots D_m^{e_m}$ with $e < e_0$ and therefore also $\omega^m e_m + \ldots + \omega e_1 + e < \omega^m e_m + \ldots + \omega e_1 + e_0$.

Once we have exhausted all D_0 what remains is some product $D_j^{e_j} \ldots D_m^{e_m}$ with $j > 0$. This product can either step by step remove some of the front variables which results in some $D_k^{e'_k} \ldots D_m^{e_m}$, where $k > j$ or $k = j$ and $e'_k < e_j$. The respective ordinal is then $\omega^m e_m + \ldots + \omega^k e'_k$ which is less than the original $\omega^m e_m + \ldots + \omega^j e_j$. Or, after a few such steps some variable D_k, $k \geq j$, performs a sequence of transitions $D_k \xrightarrow{\tau} D_k D_{k-1}$ which results in the process $D_k D_{k-1}^{e_{k-1}} D_k^{e'_k - 1} \ldots D_m^{e_m}$. Again, the respective ordinal

$\omega^m e_m + \ldots + \omega^k(e'_k - 1) + \omega^{k-1}e_{k-1}$ is less than $\omega^m e_m + \ldots + \omega^j e_j$. Then we can apply Proposition 4 to conclude that $D_k D_{k-1}^{e_{k-1}} D_k^{e'_k-1} \ldots D_m^{e_m}$ is actually weakly bisimilar to $D_k^{e'_k} \ldots D_m^{e_m}$. $\square$

4 Equivalence and Inequivalence Results

This mainly technical section concentrates on the relationship between pairs of products and individual approximants. We shall start by specifying the maximal ordinal that relates two (non-bisimilar) products. Assuming two products P and Q that are assigned ordinal numbers β and γ, respectively, P and Q will be equivalent at $\approx_\alpha$, where α is any ordinal up to the minimum of β and γ. We shall first describe on an intuitive level why this is so as the rigourous proof is rather technical. Without loss of generality, we can assume that $\alpha \leq \beta < \gamma$. That means that Q has the ability to evolve exactly into P with a $\stackrel{\tau}{\Longrightarrow}$ and hence can copy all its moves. Therefore it is the process P that needs to keep up with Q. In order to do so it will respond to moves of Q with the minimal loss of power, i.e. it will only discard the atoms necessary to match Q's moves. To demonstrate equivalence at $\approx_\alpha$, if Q performs $Q \stackrel{a}{\Longrightarrow} Q'$ then P has to be able to reach by a matching move some derivative P' such that $P' \approx_\delta Q'$, for any $\delta < \alpha$. So P will choose such a derivation that the resulting product P' will determine an ordinal $\eta \geq \delta$ (unless P can evolve directly into Q'). The possibility of such a move for P follows from Proposition 4. By finite application of these steps we reach the approximant $\approx_0$ where all processes are equivalent. Consequently, P has demonstrated equivalence with Q at level α. The precise statement is as follows:

Theorem 8 *For all products,* $\prod_{i=0}^m D_i^{e_i} \approx_\alpha \prod_{i=0}^m D_i^{f_i}$, *where* $\alpha \leq \min\{\beta, \gamma\}$ *with* $\beta = \omega^m e_m + \omega^{m-1}e_{m-1} + \ldots + \omega e_1 + e_0$ *and* $\gamma = \omega^m f_m + \omega^{m-1} f_{m-1} + \ldots + \omega f_1 + f_0$.

Proof: We will prove this statement by transfinite induction on α which consists of proving the claim for the cases of α being 0, then a successor ordinal number and finally a limit ordinal number. The claim obviously holds for $\alpha = 0$ since all processes are related at zero level.

In order to prove the successor case we assume that the claim holds for some α and we will want to prove it for $\alpha + 1$. We presuppose two processes $P \equiv \prod_{i=0}^m D_i^{e_i}$ and $Q \equiv \prod_{i=0}^m D_i^{f_i}$ such that, without loss of generality,

$\alpha + 1 \leq \beta = \sum_{i=0}^{m} \omega^i e_i \leq \gamma = \sum_{i=0}^{m} \omega^i f_i$ and we will show that $P \approx_{\alpha+1} Q$. We know that $\beta = \gamma$ if and only if $e_i = f_i$ for every $i = 0, \ldots, m$. In that case P and Q are two identical processes which are trivially equivalent at every level. Hence we shall assume that $\beta < \gamma$.

We remind ourselves that $P \approx_{\alpha+1} Q$ if for every move $P \stackrel{\mu}{\Longrightarrow} P'$ there is a matching transition $Q \stackrel{\mu}{\Longrightarrow} Q'$ with $P' \approx_\alpha Q'$, and conversely, starting from Q. Since $\sum_{i=0}^{m} \omega^i e_i < \sum_{i=0}^{m} \omega^i f_i$ the process Q can evolve into P by a $\stackrel{\tau}{\Longrightarrow}$ sequence so in case P takes the initiative and performs a transition $P \stackrel{\mu}{\Longrightarrow} P'$ the process Q will copy P and become P' as well. Then we can conclude that $P' \approx_\alpha P'$.

It remains to check the moves of $Q = \prod_{i=0}^{m} D_i^{f_i}$. Either $Q \stackrel{\mu}{\Longrightarrow} \prod_{i=0}^{m} D_i^{g_i}$, where $\sum_{i=0}^{m} \omega^i g_i \leq \gamma$, or $Q \stackrel{\mu}{\Longrightarrow} D_j \prod_{i=j-1}^{m} D_i^{g_i}$. The latter is by Proposition 4 weakly bisimilar to the product $D_j^{g_j+1} \prod_{i=j+1}^{m} D_i^{g_i}$. We can replace $D_j \prod_{i=j-1}^{m} D_i^{g_i}$ with the bisimilar product because of Corollary 5 and the following two facts: firstly, for every ordinal δ, $\approx \subseteq \approx_\delta$, and secondly, $\approx_\delta$ is transitive. Therefore if $P' \approx_\alpha D_j^{g_j+1} \prod_{i=j+1}^{m} D_i^{g_i}$ for some P-derivative P' then also $P' \approx_\alpha D_j \prod_{i=j-1}^{m} D_i^{g_i}$. Hence we will assume that Q evolves into a product $\prod_{i=0}^{m} D_i^{g_i}$. We have to distinguish two cases according to the height of the Q-derivative:

1. $\alpha < \sum_{i=0}^{m} \omega^i g_i$

   We will show that P can do a matching action and evolve into a product $\prod_{i=0}^{m} D_i^{h_i}$ with $\sum_{i=0}^{m} \omega^i h_i \geq \alpha$. Then we will use the induction hypothesis and conclude that $\prod_{i=0}^{m} D_i^{h_i} \approx_\alpha \prod_{i=0}^{m} D_i^{g_i}$. There are two ways in which P will respond depending on e_0 (the exponent of D_0 in P).

   - If $e_0 > 0$ then P contains at least one copy of D_0 which will perform the appropriate action using the transition $D_0 \xrightarrow{a/\tau} \epsilon$. P will therefore evolve into $D_0^{e_0-1} \ldots D_m^{e_m}$ with $\sum_{i=0}^{m} \omega^i e_i - 1 \geq \alpha$. Hence $\alpha \leq \min\{\sum_{i=0}^{m} \omega^i e_i - 1, \sum_{i=0}^{m} \omega^i g_i\}$ and from the induction hypothesis we can conclude that $D_0^{e_0-1} \ldots D_m^{e_m} \approx_\alpha \prod_{i=0}^{m} D_i^{g_i}$.
   - If $e_0 = 0$ then $\beta = \sum_{i=0}^{m} \omega^i e_i$ is a limit ordinal. Since $\alpha + 1$ is a successor ordinal and $\alpha + 1 \leq \beta$ then from the nature of ordinal

numbers $\alpha+1 < \beta$ and, moreover, there exists an ordinal δ with $\alpha < \delta < \beta$. Now we can use the statement of Proposition 7 and deduce that there has to be a matching move of P resulting in a product $\prod_{i=0}^{m} D_i^{h_i}$ with $\alpha < \delta = \sum_{i=0}^{m} \omega^i h_i$. Then we have again that $\alpha \leq \min\{\sum_{i=0}^{m} \omega^i h_i, \sum_{i=0}^{m} \omega^i g_i\}$ and the conclusion is that $\prod_{i=0}^{m} D_i^{h_i} \approx_\alpha \prod_{i=0}^{m} D_i^{g_i}$.

2. $\alpha \geq \sum_{i=0}^{m} \omega^i g_i$

   In this case also $\sum_{i=0}^{m} \omega^i g_i < \sum_{i=0}^{m} \omega^i e_i$ which means that by Proposition 7 the process P can simulate the move of Q and become exactly the product $\prod_{i=0}^{m} D_i^{g_i}$. Again we conclude with the argument that the relation $\approx_\alpha$ is an equivalence and so $\prod_{i=0}^{m} D_i^{g_i} \approx_\alpha \prod_{i=0}^{m} D_i^{g_i}$.

Lastly we have to check the case of a limit ordinal λ. The argument is the following: we assume that the two processes $\prod_{i=0}^{m} D_i^{e_i}$ and $\prod_{i=0}^{m} D_i^{f_i}$ are such that $\lambda \leq \min\{\sum_{i=0}^{m} \omega^i e_i, \sum_{i=0}^{m} \omega^i f_i\}$. Hence the same holds for every $\alpha < \lambda$. From the induction hypothesis we conclude that $\prod_{i=0}^{m} D_i^{e_i} \approx_\alpha \prod_{i=0}^{m} D_i^{f_i}$ for every $\alpha < \lambda$ and from the definition of a limit approximant we know that $\prod_{i=0}^{m} D_i^{e_i} \approx_\lambda \prod_{i=0}^{m} D_i^{f_i}$. $\square$

We have shown that for products P and Q and their corresponding ordinals β and γ, P is equivalent with Q at α, where α is the minimum of β and γ. In fact, α is the dividing line (for non-bisimilar products) as we shall show next that for any δ greater than α, P and Q are not equivalent at δ. Again we shall first give an informal justification of the claim. The argument is rather similar to the intuitive explanation we have presented before Theorem 8. This time it is the product that is assigned the greater ordinal of the two that takes care to retain as much height as possible. To be more precise, assuming that $\beta < \gamma$, i.e. Q is the larger product, and the level of inspection is $\delta > \alpha$, then appropriate moves of Q will be those that remain above α. From the assumptions it follows that all non-empty moves of P will always stay below α. By iterating transitions so that they satisfy this condition Q will eventually reach a stage where some of its derivatives will have $\stackrel{a}{\Longrightarrow}$ move at its disposal but no P derivative will be able to match it. Thus the inequivalence of P and Q will be sealed. The exact statement goes like this:

Theorem 9 *If* $\prod_{i=0}^{m} D_i^{e_i} \neq \prod_{i=0}^{m} D_i^{f_i}$ *then* $\prod_{i=0}^{m} D_i^{e_i} \not\approx_\alpha \prod_{i=0}^{m} D_i^{f_i}$ *where* $\alpha > \min\{\beta, \gamma\}$ *with* $\beta = \omega^m e_m + \omega^{m-1} e_{m-1} + \ldots + \omega e_1 + e_0$ *and* $\gamma = \omega^m f_m + \omega^{m-1} f_{m-1} + \ldots + \omega f_1 + f_0$.

Proof: We will again prove this statement by transfinite induction on α. For $\alpha = 0$ the statement holds vacuously. Next we check the case of a successor ordinal. We assume that the claim holds for an ordinal α and we will argue that it also holds for $\alpha + 1$. We know that $\prod_{i=0}^{m} D_i^{e_i} = \prod_{i=0}^{m} D_i^{f_i}$ if and only if $\sum_{i=0}^{m} \omega^i e_i = \sum_{i=0}^{m} \omega^i f_i$ hence without loss of generality we can presuppose two products $\prod_{i=0}^{m} D_i^{e_i}$ and $\prod_{i=0}^{m} D_i^{f_i}$ such that $\sum_{i=0}^{m} \omega^i e_i > \sum_{i=0}^{m} \omega^i f_i$ and $\alpha + 1 > \sum_{i=0}^{m} \omega^i f_i$.

Now let the larger product $\prod_{i=0}^{m} D_i^{e_i}$ take the initiative and perform $\stackrel{a}{\Longrightarrow}$ to become $\prod_{i=0}^{m} D_i^{e_i'}$ with $\sum_{i=0}^{m} \omega^i e_i' \geq \sum_{i=0}^{m} \omega^i f_i$. The possibility of such a move follows from our earlier assumption and Proposition 7. Again using the Proposition 7 and Corollary 5 we conclude that any matching move $\stackrel{a}{\Longrightarrow}$ of $\prod_{i=0}^{m} D_i^{f_i}$ will necessarily be sum decreasing, that is if $\prod_{i=0}^{m} D_i^{f_i} \stackrel{a}{\Longrightarrow} \prod_{i=0}^{m} D_i^{f_i'}$ then $\sum_{i=0}^{m} \omega^i f_i' < \sum_{i=0}^{m} \omega^i f_i$. Hence the two derivatives are distinct with $\sum_{i=0}^{m} \omega^i e_i' > \sum_{i=0}^{m} \omega^i f_i'$ and moreover, also $\alpha > \sum_{i=0}^{m} \omega^i f_i'$ and we can use the induction hypothesis to conclude that $\prod_{i=0}^{m} D_i^{e_i'} \not\approx_\alpha \prod_{i=0}^{m} D_i^{f_i'}$. Since this is true for all matching responses of $\prod_{i=0}^{m} D_i^{f_i}$, the products $\prod_{i=0}^{m} D_i^{e_i}$ and $\prod_{i=0}^{m} D_i^{f_i}$ cannot be equivalent at $\alpha + 1$.

Finally we shall assume that λ is a limit ordinal and $P \equiv \prod_{i=0}^{m} D_i^{e_i}$, $Q \equiv \prod_{i=0}^{m} D_i^{f_i}$ are distinct products such that, without loss of generality, $\sum_{i=0}^{m} \omega^i e_i > \sum_{i=0}^{m} \omega^i f_i$ and $\lambda > \sum_{i=0}^{m} \omega^i f_i$. From the definition, $P \approx_\lambda Q$ if for every $\alpha < \lambda$, $P \approx_\alpha Q$, so if there exists an $\alpha < \lambda$ with $P \not\approx_\alpha Q$ then also $P \not\approx_\lambda Q$. We have to distinguish two cases:

1. $\lambda > \sum_{i=0}^{m} \omega^i e_i$

   Then we know that there exists an ordinal α such that $\lambda > \alpha > \sum_{i=0}^{m} \omega^i e_i$. From the induction hypothesis it follows that $\prod_{i=0}^{m} D_i^{e_i} \not\approx_\alpha \prod_{i=0}^{m} D_i^{f_i}$ and we can deduce that $\prod_{i=0}^{m} D_i^{e_i} \not\approx_\lambda \prod_{i=0}^{m} D_i^{f_i}$.

2. $\sum_{i=0}^{m} \omega^i e_i \geq \lambda > \sum_{i=0}^{m} \omega^i f_i$

   In this case we can find an ordinal number α such that $\lambda > \alpha > \sum_{i=0}^{m} \omega^i f_i$. By Proposition 7 there exists a transition $\prod_{i=0}^{m} D_i^{e_i} \stackrel{a}{\Longrightarrow} \prod_{i=0}^{m} D_i^{e_i'}$

with $\alpha = \sum_{i=0}^{m} \omega^i e'_i$. For any matching move $Q \stackrel{a}{\Longrightarrow} \prod_{i=0}^{m} D_i^{f'_i}$ the ordinal number $\sum_{i=0}^{m} \omega^i f'_i$ is smaller than α. Hence we can deduce that $\prod_{i=0}^{m} D_i^{e'_i} \neq \prod_{i=0}^{m} D_i^{f'_i}$ and $\alpha > \min\{\sum_{i=0}^{m} \omega^i e'_i, \sum_{i=0}^{m} \omega^i f'_i\}$ which means that $\prod_{i=0}^{m} D_i^{e'_i} \not\approx_{\alpha} \prod_{i=0}^{m} D_i^{f'_i}$ and finally, $\prod_{i=0}^{m} D_i^{e_i} \not\approx_{\alpha+1} \prod_{i=0}^{m} D_i^{f_i}$ and $\prod_{i=0}^{m} D_i^{e_i} \not\approx_{\lambda} \prod_{i=0}^{m} D_i^{f_i}$. $\square$

5 Lower and Upper Bounds

In this section we will use the results of the previous section to deduce a lower bound on the ordinal where convergence occurs for weak bisimilarity on BPA. For a fixed n we shall define a Basic Process Algebra (Σ_n^*, Δ_n), where $\Sigma_n = \{D_0, D_1, \ldots, D_n\}$ and $\Delta_n = \{D_0 \stackrel{\tau}{\longrightarrow} \epsilon, D_0 \stackrel{a}{\longrightarrow} \epsilon, D_{i+1} \stackrel{\tau}{\longrightarrow} D_{i+1}D_i, D_{i+1} \stackrel{\tau}{\longrightarrow} \epsilon \mid 0 \leq i < n\}$. We consider the two processes D_n and D_0D_n from Σ_n^*. The ordinals assigned to D_n, resp. D_0D_n, are ω^n, resp. ω^n+1. Then by applying Theorem 8 and Theorem 9 we obtain these results:

$$D_n \approx_{\omega^n} D_0D_n \ \wedge\ D_n \not\approx_{\omega^n+1} D_0D_n.$$

Therefore we can conclude that on the algebra (Σ_n^*, Δ_n), one can distinguish the approximant $\approx_{\omega^n}$ from the maximal weak bisimulation $\approx$. This can be carried out for any n hence we come to the following conclusion:

Theorem 10 *For every $\alpha < \omega^\omega$ there exists a Basic Process Algebra (Σ^*, Δ) such that $\approx \subsetneq \approx_\alpha$ with respect to (Σ^*, Δ).*

The implication is that a lower bound on the convergence to $\approx$ is ω^ω. If we analyse the construction we can see that in order to reach higher levels we need to introduce new variables. Since we are only allowed to use a finite number of variables in the definition of a BPA this leads to the following conjecture:

Conjecture 11 *For Basic Process Algebras, $\approx = \approx_{\omega^\omega}$.*

Now we shall try to establish some upper bounds on the level of convergence. That does not seem to be so easy as we do not have appropriate tools that could establish the maximal level of convergence, even for a specific algebra. It seems that the only claim we can make stems from the fact that the process algebras we deal with are countable. We have already showed that

$$\approx_\alpha = \approx_{\alpha+1} \Longrightarrow \approx_\alpha = \approx,$$

that is if two subsequent levels α and $\alpha + 1$ define the same equivalence then all levels β for $\alpha \leq \beta$ are equal and hence equal the maximal weak bisimulation. We can define only countably many processes and hence countably many pairs of processes which means we can never distinguish more than countably many approximants. That can be expressed as follows:

Lemma 12 $\approx = \approx_{\omega_1}$.

Obviously, this is a rather crude upper bound (ω_1 is the first uncountable ordinal). Bradfield observed that there exists a stronger upper bound that can be obtained as follows. Non-bisimulation is an inductively defined property, and the monotone (and indeed positive) operator over which induction occurs is arithmetical, since the $\overset{\mu}{\Longrightarrow}$ relation for BPA is clearly arithmetical. There is a theorem due to Spector (consult Theorem IV.2.15 in [8]) that any inductive definition over a monotone arithmetical (or even Π^1_1) operator has closure ordinal $\leq \omega_1^{CK}$, the least non-recursive ordinal.

6 Discussion

To summarise the importance of the presented results we will compare strong and weak bisimulation equivalences with regard to decidability. The classical decision procedure for strong bisimilarity consists of two semidecision procedures. The algorithm for semideciding bisimilarity searches for a finite base of $\sim$. The complementary algorithm for semideciding $\not\sim$ checks all individual approximants $\sim_n$ step by step, and tests for equivalence at n.

The construction that was presented in the paper poses a serious problem to the semidecidability of $\not\approx$. Obviously enumerating approximants up to some level ω^n does not seem feasible. Moreover we may not be able to check equivalence at $\approx_{\omega^n}$. The aforementioned construction is rather simple yet it is already not clear what an appropriate method for testing (non)bisimilarity of a pair of processes would be.

On the other hand, it seems plausible that in general there might exist a finite base for the maximal weak bisimulation. Indeed it is rather easy to construct a finite base for $\approx$ for every BPA (Σ_n, Δ_n) from the presented construction. Thus we may conclude that semidecidability of $\approx$ appears plausible in contrast with semidecidability of $\not\approx$ which seems to require an entirely new technique.

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