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How to Parallelize Sequential Processes

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Abstract

A process is prime if it cannot be decomposed into a parallel product of nontrivial processes. We characterize all non-prime normed BPA processes together with their decompositions in terms of normal forms which are designed in this paper. Then we show that it is decidable whether a given normed BPA process is prime and if not, its decomposition can be effectively constructed. This brings other positive decidability results. Finally, we prove that bisimilarity is decidable in a large subclass of normed PA processes.

1 Introduction

A general problem considered by many researchers is how to improve performance of sequential programs by parallelization. In this paper we study this problem within a framework of process algebras. They provide us with a pleasant formalism which allows to specify sequential as well as parallel programs.

Here we adopt normed BPA processes as a simple model of sequential behaviours (they are equipped with a binary sequential operator). We examine the problem of effective decomposability of normed BPA processes into a parallel product of primes (a process is prime if it cannot be decomposed into nontrivial components). We design special normal forms for normed BPA processes which allow us to characterize all non-prime normed BPA processes together with their decompositions up to bisimilarity. As a consequence we also obtain a refinement of the result achieved in [BS94].

Next we show that any normed BPA process can be decomposed into a parallel product of primes effectively. We also prove several related decidability results. Finally, we prove that bisimilarity is decidable in a large subclass of normed PA processes (see [BW90]), which consists of processes of the form $\Delta_1 \| \cdots \| \Delta_n$, where each Δ_i is a normed BPA or BPP process.

In many parts of our paper we rely on results established by other researchers. The question of possible decomposition of processes into a parallel product of primes was first addressed by Milner and Moller in [MM93]. A more general result was later proved by Christensen, Hirshfeld and Moller (see [CHM93b])—it says that *each* normed process has a unique decomposition into primes up to bisimilarity. However, the proof is non-constructive.

Bisimilarity was proved to be decidable for normed BPA processes (see [BBK87, HS91, HJ94]) and normed BPP processes (see [CHM93a, HJM94]). Another property of normed BPA and BPP processes which is important for us is *regularity*. A process is regular if it is bisimilar to a process with finitely many states. Kučera proved in [Kuč96] that regularity is decidable for normed BPA and normed BPP processes in polynomial time.

The relationship between normed BPA and normed BPP processes was studied by Černá, Křetínský and Kučera in [ČKK96]. They proved that it is decidable whether for a given normed BPA (or BPP) process Δ there is some unspecified normed BPP (or BPA) process Δ' such that $\Delta \sim \Delta'$. If the answer is positive, then it is also possible to *construct* an example of such Δ'. Consequently, bisimilarity is decidable in the union of normed BPA and normed BPP processes.

2 Preliminaries

2.1 BPA and BPP processes

Let $Act = \{a, b, c, \ldots\}$ be a countably infinite set of *atomic actions*. Let $Var = \{X, Y, Z, \ldots\}$ be a countably infinite set of *variables* such that $Var \cap Act = \emptyset$. The classes of recursive BPA and BPP expressions are defined by the following abstract syntax equations:

$$\begin{array}{lcl} E_{BPA} & ::= & a \;\mid\; X \;\mid\; E_{BPA}.E_{BPA} \;\mid\; E_{BPA} + E_{BPA} \\ E_{BPP} & ::= & a \;\mid\; X \;\mid\; aE_{BPP} \;\mid\; E_{BPP} \| E_{BPP} \;\mid\; E_{BPP} + E_{BPP} \end{array}$$

Here a ranges over Act and X ranges over Var. The symbol Act^* denotes the set of all finite strings over Act and the symbol Act^+ denotes the set $Act^* - \{\epsilon\}$.

As usual, we restrict our attention to guarded expressions. A BPA or BPP expression E is *guarded* if every variable occurrence in E is within the scope of an atomic action.

A *guarded BPA (or BPP) process* is defined by a finite family Δ of recursive process equations

$$\Delta = \{X_i \stackrel{def}{=} E_i \;\mid\; 1 \leq i \leq n\}$$

where X_i are distinct elements of Var and E_i are guarded BPA (or BPP) expressions, containing variables from $\{X_1, \ldots, X_n\}$. The set of variables which appear in Δ is denoted by $Var(\Delta)$.

The variable X_1 plays a special role (X_1 is sometimes called *the leading variable*—it is a root of a labelled transition system, defined by the process Δ and following rules:

$$\frac{}{a \xrightarrow{a} \epsilon} \qquad \frac{E \xrightarrow{a} E'}{E.F \xrightarrow{a} E'.F} \qquad \frac{E \xrightarrow{a} E'}{E + F \xrightarrow{a} E'} \qquad \frac{F \xrightarrow{a} F'}{E + F \xrightarrow{a} F'}$$

$$\frac{E \xrightarrow{a} E'}{E\|F \xrightarrow{a} E'\|F} \qquad \frac{F \xrightarrow{a} F'}{E\|F \xrightarrow{a} E\|F'} \qquad \frac{E \xrightarrow{a} E'}{X \xrightarrow{a} E'}\ (X \stackrel{def}{=} E \in \Delta)$$

The symbol ϵ denotes the empty expression. Presented rules should be considered modulo *structural congruence*, which is the smallest congruence relation over BPA and BPP expressions such that the following laws hold:

- associativity and 'ϵ' as a unit for sequential composition (the '.' operator).

- associativity, commutativity and 'ϵ' as a unit for parallel composition (the '$\|$' operator).

- associativity, commutativity and 'ϵ' as a unit for nondeterministic choice (the '$+$' operator).

Nodes of the transition system generated by Δ are BPA (or BPP) expressions, which are often called *states of* Δ, or just "states" when Δ is understood from the context. We also define the relation $\xrightarrow{w}^*$, where $w \in Act^*$, as the reflexive and transitive closure of $\xrightarrow{a}$ (we often write $E \rightarrow^* F$ instead of $E \xrightarrow{w}^* F$ if w is irrelevant). Given two states E, F, we say that F is *reachable from* E, if $E \rightarrow^* F$. States of Δ which are reachable from X_1 are said to be *reachable*.

Remark 1. *Processes are often identified with their leading variables. Furthermore, if we assume fixed processes Δ_1, Δ_2 such that $Var(\Delta_1) \cap Var(\Delta_2) = \emptyset$, then we can view any process expression E (not necessarily guarded) whose variables are defined in Δ_1, Δ_2 as a process too—if we denote this process by Δ, then the leading equation of Δ is $X \stackrel{def}{=} E'$, where $X \notin Var(\Delta_1) \cup Var(\Delta_2)$ and E' is a process expression which is obtained from E by substituting each variable in E with the right-hand side of its corresponding defining equation in Δ_1 or Δ_2 (E' must be guarded now). Moreover, def. equations from Δ_1, Δ_2 are added to Δ. All notions originally defined for processes can be used for process expressions in this sense too.*

2.1.1 Bisimulation

The equivalence between process expressions (states) we are interested in here is *bisimilarity* [Par81], defined as follows:

Definition 1. *A binary relation R over process expressions is a* bisimulation *if whenever $(E, F) \in R$ then for each $a \in Act$*

- *if $E \xrightarrow{a} E'$, then $F \xrightarrow{a} F'$ for some F' such that $(E', F') \in R$*
- *if $F \xrightarrow{a} F'$, then $E \xrightarrow{a} E'$ for some E' such that $(E', F') \in R$*

Processes Δ and Δ' are bisimilar, *written $\Delta \sim \Delta'$, if their leading variables are related by some bisimulation.*

2.1.2 Normed processes

An important subclass of BPA and BPP processes can be obtained by an extra restriction of *normedness*. A variable $X \in Var(\Delta)$ is *normed* if there is $w \in Act^*$ such that $X \xrightarrow{w}^* \epsilon$. In that case we define the *norm* of X, written

$|X|$, to be the length of the shortest such w. A process Δ is *normed* if all variables of $Var(\Delta)$ are normed. The norm of Δ is then defined to be the norm of X_1.

Remark 2. *As normed processes are intensively studied in this paper, we emphasize some properties of the norm:*

- *Note the norm of a normed process is easy to compute:* $|a| = 1$, $|E + F| = \min\{|E|, |F|\}$, $|E.F| = |E| + |F|$, $|E \| F| = |E| + |F|$ *and if* $X_i \stackrel{def}{=} E_i$ *and* $|E_i| = n$, *then* $|X_i| = n$.
- *Bisimilar processes must have the same norm.*

2.1.3 Greibach normal form

Any BPA or BPP process Δ can be effectively presented in a special normal form which is called 3-Greibach normal form by analogy with CF grammars (see [BBK87] and [Chr93]). Before the definition we need to introduce the set $Var(\Delta)^*$ of all finite sequences of variables from $Var(\Delta)$, and the set $Var(\Delta)^{\otimes}$ of all finite multisets over $Var(\Delta)$. Each multiset of $Var(\Delta)^{\otimes}$ denotes a BPP expression by combining its elements in parallel using the '$\|$' operator.

Definition 2. *A BPA (or BPP) process* Δ *is said to be in* Greibach normal form (GNF) *if all its equations are of the form*

$$X \stackrel{def}{=} \sum_{j=1}^{n} a_j \alpha_j$$

where $n \in N$, $a_j \in Act$ *and* $\alpha_j \in Var(\Delta)^*$ *(or* $\alpha_j \in Var(\Delta)^{\otimes}$*). We also require that each* $Y \in Var(\Delta)$ *appears in some reachable state of* Δ*. If* $Length(\alpha_j) \leq 2$ *(or* $card(\alpha_j) \leq 2$*) for each* j, $1 \leq j \leq n$, *then* Δ *is said to be in 3-GNF.*

From now on we assume that all BPA and BPP processes we are working with are presented in GNF. This justifies also the assumption that all reachable states of a BPA process Δ are elements of $Var(\Delta)^*$ and all reachable states of a BPP process Δ' are elements of $Var(\Delta')^{\otimes}$.

2.2 Regular processes

Many proofs in this paper take advantage of the fact that regularity of normed BPA and normed BPP processes is decidable (even in polynomial time—see [Kuč96]). The next definition explains what is meant by the notion of regularity and introduce standard normal form for regular processes.

Definition 3. *A process Δ is* regular *if there is a process Δ' with finitely many states such that $\Delta \sim \Delta'$. A regular process Δ is said to be in normal form if all its equations are of the form*

$$X \stackrel{def}{=} \sum_{j=1}^{n} a_j X_j$$

where $n \in N$, $a_j \in Act$ and $X_j \in Var(\Delta)$.

It is easy to see that a process is regular iff it can reach only finitely many states up to bisimilarity. In [Mil89] it is shown, that regular processes can be represented in the normal form just defined. Thus a process Δ is regular iff there is a regular process Δ' in normal form such that $\Delta \sim \Delta'$. A proof of the following proposition can be found in [Kuč96].

Proposition 1. *Let Δ be a normed BPA or BPP process. The problem whether Δ is regular is decidable in polynomial time. Moreover, if Δ is regular then a regular process Δ' in normal form such that $\Delta \sim \Delta'$ can be effectively constructed.*

2.3 Special notation

In the rest of this paper we also use some special notation (due to the lack of general standard). To improve readability of our paper we put all specialties to one place:

- **nBPA** and **nBPP** are abbreviations for normed BPA and normed BPP, respectively.
- if α is a state of a nBPA or nBPP process such that α is regular (see Remark 1), then $\boldsymbol{\Delta}^{\mathcal{R}}(\alpha)$ denotes a bisimilar regular process in normal form, which can be effectively constructed due to Proposition 1. Furthermore, we always assume that $\Delta^{\mathcal{R}}(\alpha)$ contains completely fresh variables which are not contained in any other process we deal with.

- the class of all processes for which there is a bisimilar nBPA (or nBPP) process is denoted $\mathcal{S}$*(nBPA)* (or $\mathcal{S}$*(nBPP)*).
- if $\Delta_1, \ldots, \Delta_n$ are processes from nBPA $\cup$ nBPP and X_i is the leading variable of Δ_i for $1 \leq i \leq n$, then $\Delta_1 \| \cdots \| \Delta_n$ denotes the process $X_1 \| \cdots \| X_n$ in the sense of Remark 1.
- square brackets '[' and ']' indicate optional occurrence—if we say that some expression is of the form $a[A][B]$, we mean that this expression is either a, aA, aB or aAB.
- upper indexes are used heavily; they appear in two forms:

$$\begin{aligned}
\alpha^i &= \underbrace{\alpha \| \cdots \| \alpha}_{i} \\
\alpha^{\bullet i} &= \underbrace{\alpha . \cdots . \alpha}_{i}
\end{aligned}$$

2.4 Decidability of bisimilarity in nBPA $\cup$ nBPP

Bisimilarity is known to be decidable for nBPA (see [BBK87, HS91, HJ94]) and nBPP (see [CHM93a, HJM94]) processes. The following result due to Černá, Křetínský and Kučera (see [ČKK96]) says that bisimilarity is decidable even in the union of nBPA and nBPP processes.

Proposition 2. *Let Δ be a nBPA (or nBPP) process. It is decidable, whether $\Delta \in \mathcal{S}$(nBPP) (or whether $\Delta \in \mathcal{S}$(nBPA)) and if the answer is positive, then a bisimilar nBPP (or nBPA) process can be effectively constructed.*

2.5 Decomposability, prime processes

Definition 4 (prime processes). *Let* nil *be a special name for the process which cannot emit any action (i.e.,* nil $\sim \epsilon$*). A nBPA or nBPP process Δ is* prime *if* $\Delta \not\sim$ nil *and whenever $\Delta \sim \Delta_1 \| \Delta_2$ we have that either $\Delta_1 \sim$* nil *or $\Delta_2 \sim$* nil*.*

Natural questions are, what processes have a decomposition into a finite parallel product of primes and whether this decomposition is unique. This problem was first examined by Milner and Moller in [MM93]. They proved that each normed finite process has a unique decomposition up to bisimilarity. A more general result is due to Christensen, Hirshfeld and Moller—they proved the following proposition (see [CHM93b]):

Proposition 3. *Let Δ be a nBPP process. Then Δ has a unique decomposition (up to bisimilarity) into a parallel product of primes.*

Remark 3. *Proposition 3 in fact holds for* any *normed process (namely for nBPA). The proof does not depend on a concrete syntax—it could be easily formulated in terms of normed transition systems.*

Proposition 3 in fact says that each normed process Δ can be parallelized in the "best" way and that this way is in some sense unique. However, this nice theoretical result is non-constructive. It is not clear how to *construct* the decomposition and how to test whether some process is prime. This is the subject of next sections.

3 Decomposability of nBPP processes

Each nBPP processes Δ can be easily decomposed into a parallel product of primes—all what has to be done is a construction of a bisimilar *canonical* process (see [Chr93]).

Theorem 1. *Let Δ be a nBPP process. It is decidable whether Δ is prime and if not, its decomposition into primes can be effectively constructed.*

Proof: By induction on $n = |\Delta|$:

- **n=1:** each nBPP process whose norm is 1 is prime.
- **Induction step:** Suppose $\Delta \sim \Delta_1 \| \Delta_2$. As Δ_1, Δ_2 are reachable states of $\Delta_1 \| \Delta_2$, there are $\alpha_1, \alpha_2 \in Var(\Delta)^{\otimes}$ such that $\Delta_1 \sim \alpha_1$ and $\Delta_2 \sim \alpha_2$, thus $\Delta \sim \alpha_1 \| \alpha_2$. Furthermore, $|\Delta| = |\alpha_1| + |\alpha_2|$. We show that there are only finitely many candidates for α_1, α_2. First, there are only finitely many pairs $[k_1, k_2] \in N \times N$ such that $k_1 + k_2 = |\Delta|$. For each such pair $[k_1, k_2]$ there are only finitely many pairs $[\beta_1, \beta_2]$ such that $\beta_1, \beta_2 \in Var(\Delta)^{\otimes}$, $|\beta_1| = k_1$ and $|\beta_2| = k_2$. It is obvious that the set $\mathcal{M}$ of all such pairs can be effectively constructed. For each element $[\beta_1, \beta_2]$ of $\mathcal{M}$ we check whether $\Delta \sim \beta_1 \| \beta_2$ (it can be done because bisimilarity is decidable for nBPP processes). If there is no such pair

then Δ is prime. Otherwise, we check whether β_1, β_2 are prime (it is possible by ind. hypothesis) and construct their decompositions. If we put obtained decompositions in parallel, we get a decomposition of Δ. $\square$

As each normed regular process in normal form can be seen as a nBPP process in GNF, Theorem 1 (and especially its constructive proof) can be used also for regular nBPA processes (see Proposition 1). In the next section we can thus concentrate on non-regular nBPA processes.

4 Decomposability of nBPA processes

It this section we give an exact characterization of non-prime nBPA processes. We design special normal forms which allow us to characterize all non-prime nBPA processes together with their decompositions (up to bisimilarity). Our results bring also interesting consequences—we obtain a refinement of the result achieved in [BS94] (see Remark 6) and we also show that any nBPA process can be decomposed into prime processes effectively. Further positive decidability results are discussed in the end of the second subsection. Finally, we also prove that bisimilarity is decidable in a natural subclass of normed PA processes.

4.1 Normal forms for non-prime nBPA processes

In this subsection we design the promised normal forms both for non-prime nBPA processes and prime processes which appear in corresponding decompositions. As we already know from the previous section, the problem of possible decomposition of a nBPA process into a parallel product of primes is actually interesting only for non-regular nBPA processes, hence the main characterization theorem does not concern regular nBPA processes.

The layout of this subsection is as follows: first we prove two technical lemmas (Lemma 1 and 2). Then we consider the following problem: if Δ is a non-regular nBPA process such that $\Delta \sim \Delta_1 \| \Delta_2$, where Δ_1, Δ_2 are some (unspecified) processes, how do the processes $\Delta, \Delta_1, \Delta_2$ look like? It is clear that $\Delta_1, \Delta_2 \in \mathcal{S}(nBPA)$, hence the assumption that Δ_1, Δ_2 are nBPA processes can be used w.l.o.g. This problem is solved by Proposition 4 and 5, with a help of several definitions. Having this, the proof of Theorem 2 is easy to complete.

Lemma 1. *Let Δ be a nBPA process. Let $\alpha, \gamma \in Var(\Delta)^+$, $Q, C \in Var(\Delta)$ such that $|Q| = |C| = 1$ and $\alpha \| Q \sim C.\gamma$. Then $\alpha \sim Q^{|\alpha|}$.*

Proof: It suffices to prove that if $\beta \| Q^i \sim C.\gamma$ where $\beta \in Var(\Delta)^+$ and $i \in N$, then $\beta \| Q^i \sim \beta' \| Q^{i+1}$ for some $\beta' \in Var(\Delta)^*$. As $|C| = 1$, all states which are reachable from $\beta \| Q^i$ in one norm-decreasing step are bisimilar. As Δ is normed, $\beta \xrightarrow{a} \beta'$ where $|\beta| = |\beta'| + 1$ and $a \in Act$. Hence $\beta \| Q^{i-1} \sim \beta' \| Q^i$ and by substitution we obtain $\beta \| Q^i \sim \beta' \| Q^{i+1}$. □

The proof of the following lemma is probably the most technical part of our paper. Diagrams on Figure 1 could ease the reading.

Lemma 2. *Let Δ be a nBPA process, $\alpha, \beta, \gamma \in Var(\Delta)^*$ such that α is non-regular and $\alpha \| \beta \sim \gamma$. Let $\beta \rightarrow^* Q$ where $|Q| = 1$. Then $\beta \sim Q^{|\beta|}$.*

Proof: As α is non-regular, it can reach a state of an arbitrary length, i.e., for each $i \in N$ there is α' such that $\alpha \rightarrow^* \alpha'$ and $Length(\alpha') = i$. Let $m = \max\{|X|, X \in Var(\Delta)\}$ and let $\alpha \rightarrow^* \alpha_1$ where $Length(\alpha_1) \geq m.(|\beta|+1)$. Then $\alpha_1 \| \beta \sim \gamma_1$ for some $\gamma_1 \in Var(\Delta)^*$. As $\beta \rightarrow^* Q$, $\alpha_1 \| Q \sim \gamma_2$ where $\gamma_2 \in Var(\Delta)^*$ and $Length(\gamma_2) > 1$ — hence γ_2 is of the form $P.\omega$ where $\omega \in Var(\Delta)^+$. Let $\alpha_1 \xrightarrow{s}{}^* \alpha_2$ where s is a norm-decreasing sequence of actions such that $Length(s) = |P| - 1$. As $\alpha_1 \| Q \xrightarrow{s}{}^* \alpha_2 \| Q$ and $\alpha_1 \| Q \sim P.\omega$, $P.\omega \xrightarrow{s}{}^* C.\omega$ where $|C| = 1$ and $\alpha_2 \| Q \sim C.\omega$. Now we can apply Lemma 1 and conclude $\alpha_2 \sim Q^{|\alpha_2|}$. As $\alpha_1 \xrightarrow{s}{}^* \alpha_2$ where $Length(s) = |P| - 1 < m$, only the first $m - 1$ variables of α_1 could contribute to the sequence s — hence α_1, α_2 must have a common suffix whose length is at least $m.|\beta|$, i.e., $\alpha_1 = \nu.\eta$, $\alpha_2 = \delta.\eta$ where $Length(\eta) \geq m.|\beta|$. As $\alpha_1 \| \beta \sim \gamma_1$ and $\alpha_1 = \nu.\eta$, we can conclude $\eta \| \beta \sim \gamma_3$ for some $\gamma_3 \in Var(\Delta)^*$. Clearly $Length(\gamma_3) > |\beta|$, because $Length(\eta) \geq m.|\beta|$ (and thus also $|\eta| \geq m.|\beta|$) and therefore $|\eta \| \beta| > m.|\beta|$. Thus γ_3 is of the form $A_1. \cdots .A_{|\beta|+1}.\rho$ where $\rho \in Var(\Delta)^*$. Furthermore, $\eta \sim Q^{|\eta|}$ because $\alpha_2 \sim Q^{|\alpha_2|}$ and $\alpha_2 = \delta.\eta$. To sum up, we have $Q^{|\eta|} \| \beta \sim A_1. \cdots .A_{|\beta|+1}.\rho$. Now we prove that $\beta \sim Q^{|\beta|}$. Let $\beta \xrightarrow{t}{}^* \epsilon$ where $Length(t) = |\beta|$. Then $Q^{|\eta|} \| \beta \xrightarrow{t}{}^* Q^{|\eta|}$ and the state $A_1. \cdots .A_{|\beta|+1}.\rho$ must be able to match the sequence t and enter a state bisimilar to $Q^{|\eta|}$. As

$$
\begin{tikzcd}
\alpha\|\beta \arrow[d] & \sim & \gamma \arrow[d] \\
\alpha_1\|\beta \arrow[d] & \sim & \gamma_1 \arrow[d] \\
\alpha_1\|Q \arrow[d, "s"] & \sim & \gamma_2 = P.\omega \arrow[d, "s"] \\
\alpha_2\|Q & \sim & C.\omega
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & Q^{|\eta|}\|\beta \arrow[d, "t"] \arrow[ddll, "v"'] & \sim & A_1.\cdots.A_{|\beta|+1}.\rho \arrow[d, "t"] \arrow[dd, bend left, "v"] \\
 & & Q^{|\eta|} \arrow[d, "u"] & \sim & \varphi.A_{|\beta|+1}.\rho \arrow[d, "u"] \\
Q^{|\eta|-|A_1.\cdots.A_{|\beta|}|}\|\beta & \sim & Q^{|\eta|-|\varphi|} & \sim & A_{|\beta|+1}.\rho
\end{tikzcd}
$$

Figure 1: Diagrams for the proof of Lemma 2

$Length(t) = |\beta|$, only the first $|\beta|$ variables of $A_1.\cdots.A_{|\beta|+1}.\rho$ can contribute to the sequence t, i.e., $A_1.\cdots.A_{|\beta|+1}.\rho \xrightarrow{t}^* \varphi.A_{|\beta|+1}.\rho$ where $\varphi \in Var(\Delta)^*$. Now let $\varphi.A_{|\beta|+1}.\rho \xrightarrow{u}^* A_{|\beta|+1}.\rho$ where $Length(u) = |\varphi|$. The state $Q^{|\eta|}$ can match the sequence u only by removing $|\varphi|$ copies of Q — hence $Q^{|\eta|-|\varphi|} \sim A_{|\beta|+1}.\rho$. As $|\eta| \geq m.|\beta|$, it is clear that $|\eta| \geq |A_1.\cdots.A_{|\beta|}|$. Therefore there is $v \in Act^*$, $Length(v) = |A_1.\cdots.A_{|\beta|}|$ such that $Q^{|\eta|} \xrightarrow{v}^* Q^{|\eta|-|A_1.\cdots.A_{|\beta|}|}$ and thus $Q^{|\eta|}\|\beta \xrightarrow{v}^* Q^{|\eta|-|A_1.\cdots.A_{|\beta|}|}\|\beta$. The state $A_1.\cdots.A_{|\beta|+1}.\rho$ can match the sequence v only by removing $A_1.\cdots.A_{|\beta|}$ — hence $Q^{|\eta|-|A_1.\cdots.A_{|\beta|}|}\|\beta \sim A_{|\beta|+1}.\rho$ and by transitivity of bisimilarity we have $Q^{|\eta|-|\varphi|} \sim Q^{|\eta|-|A_1.\cdots.A_{|\beta|}|}\|\beta$. From this we obtain $\beta \sim Q^{|\beta|}$. $\square$

Definition 5 (simple processes). *A nBPA process* Δ *is* simple *if* $Var(\Delta)$ *contains just one variable, i.e.,* $card(Var(\Delta)) = 1$.

We will often identify simple processes with their leading (and only) variables in the rest of this paper. Moreover, it is easy to see that a simple process Q is non-regular iff the def. equation for Q contains a summand of the form $aQ^{\bullet k}$ where $a \in Act$ and $k \geq 2$. The norm of Q is one, because Q could not be normed otherwise. Another important property of simple processes is presented in the remark below:

Remark 4. *Each simple nBPA process Q belongs to* $\mathcal{S}$*(nBPP)—a bisimilar nBPP process can be obtained just by replacing the '.' operator with '*$\|$*' operator in the def. equation for Q. Consequently, any process expressions built over k copies of Q using '.' and '*$\|$*' operators are bisimilar (e.g.,* $(Q.(Q\|Q))\|Q \sim (Q\|Q).(Q\|Q)$*).*

Proposition 4. *Let Δ_1, Δ_2 be non-regular nBPA processes. Then $\Delta_1 \| \Delta_2 \in \mathcal{S}$(nBPA) iff $\Delta_1 \sim Q^{|\Delta_1|}$ and $\Delta_2 \sim Q^{|\Delta_2|}$ for some non-regular simple process Q.*

Proof:
"$\Leftarrow$" Easy—see Remark 4.
"$\Rightarrow$" Assume there is some nBPA process Δ such that $\Delta_1 \| \Delta_2 \sim \Delta$. Then there are $\alpha_1, \alpha_2 \in Var(\Delta)^*$ such that $\Delta_1 \sim \alpha_1$ and $\Delta_2 \sim \alpha_2$. Thus $\alpha_1 \| \alpha_2 \sim \Delta$ and as α_1, α_2 are non-regular, we can use Lemma 2 and conclude that there are $Q_1, Q_2 \in Var(\Delta)$ such that $|Q_1| = |Q_2| = 1$, $\alpha_1 \to^* Q_1$, $\alpha_2 \to^* Q_2$ and $\alpha_1 \sim Q_1^{|\alpha_1|}$, $\alpha_2 \sim Q_2^{|\alpha_2|}$. First we prove that $Q_1 \sim Q$ for some simple process Q. To do this, it suffices to prove that if $a\gamma$ is a summand in the def. equation for Q_1, then $\gamma \sim Q_1^{\bullet|\gamma|}$. As $\alpha_1 \| \alpha_2 \to^* Q_1 \| \alpha_2 \xrightarrow{a} \gamma \| \alpha_2$, the process $\gamma \| \alpha_2$ belongs to $\mathcal{S}$(nBPA). Let $\gamma \to^* R$ where $|R| = 1$. Then $\gamma \sim R^{|\gamma|}$ (due to Lemma 2) and as $\alpha_1 \to^* \gamma \to^* R$, we also have $\alpha_1 \sim R^{|\alpha_1|}$. Hence $R \sim Q_1$ and $\gamma \sim Q_1^{|\gamma|} \sim Q_1^{\bullet|\gamma|}$.

To finish the proof we need to show that $Q_1 \sim Q_2$. Let $m = \max\{|X|, X \in Var(\Delta)\}$. As α_1 is non-regular, it can reach a state of an arbitrary norm—let $\alpha_1 \to^* \alpha_1'$ where $|\alpha_1'| = m$. Then $\alpha_1' \| Q_2 \sim \delta$ for some $\delta \in Var(\Delta)^*$ whose length is at least two—$\delta = A.B.\delta'$. Clearly $\alpha_1' \sim Q_1^{|\alpha_1'|}$ (we can use the same argument as in the first part of this proof—Q_2 is non-regular and α' plays the role of γ), hence $Q_1^{|\alpha_1'|} \| Q_2 \sim A.B.\delta'$. As $Q_1^{|\alpha_1'|-|A|} \| Q_2 \sim B.\delta'$ and $Q_1^{|\alpha_1'|-|A|+1} \sim B.\delta'$, we have $Q_1^{|\alpha_1'|-|A|} \| Q_2 \sim Q_1^{|\alpha_1'|-|A|+1}$ by transitivity and thus $Q_1 \sim Q_2$. $\square$

Proposition 4 in fact says that if Δ is a non-regular nBPA process such that $\Delta \sim \Delta_1 \| \Delta_2$, where Δ_1, Δ_2 are non-regular processes, then each of those three processes can be equivalently represented as a power of some non-regular simple process. This representation is very special and can be seen as normal form.

If Δ is a non-regular nBPA process such that $\Delta \sim \Delta_1 \| \Delta_2$, it is also possible that Δ_1 is non-regular and Δ_2 regular. Before we start to examine this possibility, we introduce a special normal form for nBPA processes (as we shall see, Δ and Δ_1 can be represented in this normal form):

Definition 6 (DNF(Q)). *Let Δ be a non-regular nBPA process in GNF, $Q \in Var(\Delta)$. We say that Δ is in DNF(Q) if all summands in all defining equations from Δ are of the form $a([Y].[Q^{\bullet i}])$, where $Y \in Var(\Delta)$, $i \in N$ and $a \in Act$.*

Furthermore, all summands in the def. equation for Q must be of the form a[Q], where $a \in Act$.

Example 1. *The following process is in $DNF(Q)$:*

$$\begin{aligned} X &\stackrel{def}{=} a(Y.Q.Q) + bX + a(Q.Q.Q) + c \\ Y &\stackrel{def}{=} bQ + cX + c(Y.Q) + b \\ Q &\stackrel{def}{=} aQ + bQ + a + c \end{aligned}$$

Remark 5. *Reachable states of a process Δ in $DNF(Q)$ are of the form $[Y].[Q^{\bullet i}]$ where $Y \in Var(\Delta)$ and $i \in N \cup \{0\}$. As Δ is non-regular, the state $Q^{\bullet k}$ is reachable for each $k \in N$.*

Note that the variable Q itself is a regular simple process. The next lemma says that if Δ is a process in $DNF(Q)$, then the variable Q is in some sense unique:

Lemma 3. *Let Δ and Δ' be processes in $DNF(Q)$ and $DNF(R)$, respectively. If $\Delta \sim \Delta'$, then $Q \sim R$.*

Proof: Let $m = \max\{|X|,\ X \in Var(\Delta')\}$. As the state $Q^{\bullet m+1}$ is a reachable state of Δ, $Q^{\bullet m+1} \sim [Y].R^{\bullet i}$ for some $Y \in Var(\Delta')$, $i \in N$ (see Remark 5). Hence $Q \sim R$. $\square$

Proposition 5. *Let Δ_1, Δ_2 be nBPA processes such that Δ_1 is non-regular and Δ_2 is regular. Then $\Delta_1 \| \Delta_2 \in \mathcal{S}(nBPA)$ iff there is a process Δ_1' in $DNF(Q)$ such that $\Delta_1 \sim \Delta_1'$ and $\Delta_2 \sim Q^{|\Delta_2|}$.*

Proof:
"$\Rightarrow$" Let $\Delta_2 \rightarrow^* Q'$ where $Q' \in Var(\Delta_2)$, $|Q'| = 1$. Using the same kind of argument as in the proof of Proposition 4 we obtain that $Q' \sim Q$ for some regular simple process Q such that $\Delta_2 \sim Q^{|\Delta_2|}$. It remains to prove that there is a process Δ_1' in $DNF(Q)$ such that $\Delta_1 \sim \Delta_1'$. We show that each summand of each defining equation from Δ_1 can be transformed into a form which is admitted by $DNF(Q)$. First, let us realize two facts about summands—if $a\alpha$ is a summand in a def. equation from Δ_1, then

1. If $\alpha = \beta.Y.\gamma$ where Y is a non-regular variable, then each variable P of γ is bisimilar to $Q^{|P|}$.

2. α contains at most one non-regular variable.

The first fact is a consequence of Lemma 1—let Δ be a nBPA process such that $\Delta_1 \| \Delta_2 \sim \Delta$. As Δ_1 is normed, $\Delta_1 \rightarrow^* Y.\gamma.\delta$ for some $\delta \in Var(\Delta_1)^*$. As Y is non-regular, it can reach a state of an arbitrary length—let $m = \max\{|X|,\ X \in Var(\Delta_1)\}$ and let $Y \rightarrow^* \omega$ where $Length(\omega) = m$. As $\Delta_1 \| \Delta_2 \rightarrow^* \omega.\gamma.\delta \| Q'$, there is $\varphi \in Var(\Delta)^*$ such that $\omega.\gamma.\delta \| Q' \sim \varphi$. Let $\varphi = C.\varphi'$ and let s be a norm-decreasing sequence of actions such that $Length(s) = |C| - 1$ and $\omega \xrightarrow{s}{}^* \omega'$. Then $\omega'.\gamma.\delta \| Q' \sim C'.\varphi'$ where $|C'| = 1$ and due to Lemma 1 (and the fact that $Q' \sim Q$) we have $\omega'.\gamma.\delta \sim Q^{|\omega'.\gamma.\delta|}$, hence $\gamma \sim Q^{|\gamma|}$ and $P \sim Q^{|P|}$ for each variable P which appears in γ.

The second fact is a consequence of the first one—assume that $\alpha = \beta.Y.\gamma.Z.\delta$ where Y, Z are non-regular. Then $Z \sim Q^{|Z|}$ and as Q is regular, $Q^{|Z|}$ is regular too. Hence Z is regular and we have a contradiction.

Now we can describe the promised transformation of Δ_1 into Δ_1': if $X \stackrel{def}{=} \sum_{i=1}^{n} a_i \alpha_i$ is a def. equation in Δ_1, then $X \stackrel{def}{=} \sum_{i=1}^{n} a_i \mathcal{T}(\alpha_i)$ is a def. equation in Δ_1', where $\mathcal{T}$ is defined as follows:

- If α_i does not contain any non-regular variable, then $\mathcal{T}(\alpha_i) = A$, where A is the leading variable of $\Delta^{\mathcal{R}}(\alpha_i)$. Moreover, defining equations of $\Delta^{\mathcal{R}}(\alpha_i)$ are added to Δ_1'.

- If $\alpha_i = \beta.Y.\gamma$ where Y is a non-regular variable, then $\mathcal{T}(\alpha_i) = A$, where A is the leading variable of the process Δ' which is obtained by the following modification of the process $\Delta^{\mathcal{R}}(\beta)$: each summand in each def. equation of $\Delta^{\mathcal{R}}(\beta)$ which is of the form b, where $b \in Act$, is replaced with $b(Y.Q^{\bullet|\gamma|})$ — remember $B \sim Q^{|\gamma|} \sim Q^{\bullet|\gamma|}$. Moreover, def. equations of Δ' are added to Δ_1'.

The defining equation for Q is also added to Δ_1'. The resulting process is in $DNF(Q)$ and as $\mathcal{T}$ preserves bisimilarity, $\Delta_1 \sim \Delta_1'$.

"$\Leftarrow$" We show how to construct a nBPA process Δ which is bisimilar to $\Delta_1' \| Q^{|\Delta_2|}$. Let $k = |\Delta_2|$. The set of variables of Δ looks as follows:

$$Var(\Delta) = \{Q\} \cup \{Y_i,\ Y \in Var(\Delta_1'), Y \neq Q \text{ and } i \in \{0, \ldots, k\}\}$$

Defining equations of Δ are constructed using following rules:

- the def. equation for Q is the same as in Δ_1'

- if $a(Y.Q^j)$, where $j \in N \cup \{0\}$, $Y \neq Q$, is a summand in the def. equation for $Z \in Var(\Delta'_1)$, then $a(Y_i.Q^j)$ is a summand in the def. equation for Z_i for each $i \in \{0, \ldots, k\}$

- if $a(Q^j)$ where $j \in N \cup \{0\}$ is a summand in the def. equation for $Z \in Var(\Delta'_1)$, then $a(Q^{j+i})$ is a summand in the def. equation for Z_i for each $i \in \{0, \ldots, k\}$

- if aQ is a summand in the def. equation for Q and $Z \in Var(\Delta'_1)$, $Z \neq Q$, then aZ_i is a summand in the def. equation for Z_i for each $i \in \{1, \ldots, k\}$

- if a is a summand in the def. equation for Q and $Z \in Var(\Delta'_1)$, $Z \neq Q$, then aZ_{i-1} is a summand in the def. equation for Z_i for each $i \in \{1, \ldots, k\}$

The intuition which stands behind this construction is that lower indexes of variables indicate how many copies of Q in $Q^{|\Delta_2|}$ have not disappeared yet. The fact $\Delta'_1 \| Q^{|\Delta_2|} \sim \Delta$ is easy to check. $\square$

Example 2. *If we apply the algorithm presented in the "$\Leftarrow$" part of the proof of Proposition 5 to the process $X \| Q^2$, where X, Q are variables of the process presented in Example 1, we obtain the following output:*

$$
\begin{aligned}
X_2 &\stackrel{def}{=} a(Y_2.Q.Q) + bX_2 + a(Q.Q.Q.Q.Q) + c(Q.Q) + aX_2 + bX_2 + aX_1 + cX_1 \\
X_1 &\stackrel{def}{=} a(Y_1.Q.Q) + bX_1 + a(Q.Q.Q.Q) + cQ + aX_1 + bX_1 + aX_0 + cX_0 \\
X_0 &\stackrel{def}{=} a(Y_0.Q.Q) + bX_0 + a(Q.Q.Q) + c \\
Y_2 &\stackrel{def}{=} b(Q.Q.Q) + cX_2 + c(Y_2.Q) + b(Q.Q) + aY_2 + bY_2 + aY_1 + cY_1 \\
Y_1 &\stackrel{def}{=} b(Q.Q) + cX_1 + c(Y_1.Q) + bQ + aY_1 + bY_1 + aY_0 + cY_0 \\
Y_0 &\stackrel{def}{=} bQ + cX_0 + c(Y_0.Q) + b \\
Q &\stackrel{def}{=} aQ + bQ + a + c
\end{aligned}
$$

Remark 6. *Proposition 5 can also be seen as a refinement of the result achieved in [BS94]—Burkart and Steffen proved that PDA processes are closed under parallel composition with finite-state processes, while BPA processes lack this property. Proposition 5 says precisely, which nBPA processes can remain nBPA if they are put in parallel with a regular process. Moreover, it also characterizes all such regular processes.*

It is easy to see that the algorithm from the proof of Proposition 5 always outputs a process in $DNF(Q)$ (see Example 2). Moreover, the structure of this process is very specific; we can observe that each variable belongs to a special "level". This intuition is formally expressed by the following definition (it is a little complicated—but it pays because we will be able to characterize all non-prime nBPA processes):

Definition 7. *Let Δ be a nBPA process in $DNF(Q)$. The* level *of Δ, denoted $Level(\Delta)$, is the maximal $l \in N$ such that the set $Var(\Delta) - \{Q\}$ can be divided into l disjoint linearly ordered subsets $L_1, \ldots, L_l$ of the same cardinality k. Moreover, the following conditions must be true (the j^{th} element of L_i is denoted $A_{i,j}$):*

- $A_{l,1}$ is the leading variable of Δ.
- Defining equations for variables of L_1 contain only variables from $L_1 \cup \{Q\}$
- The defining equation for $A_{i,j}$, where $i \geq 2, 1 \leq j \leq k$, contains exactly those summands which can be derived by one of the following rules:
    1. If aQ is a summand in the defining equation for Q, then $aA_{i,j}$ is a summand in the defining equation for $A_{i,j}$ for each $2 \leq i \leq l$, $1 \leq j \leq k$.
    2. If a is a summand in the defining equation for Q, then $aA_{i-1,j}$ is a summand in the defining equation for $A_{i,j}$ for each $2 \leq i \leq l$, $1 \leq j \leq k$.
    3. If $a(A_{1,m}.Q^{\bullet n})$ is a summand in the defining equation for $A_{1,j}$ such that $A_{1,m} \neq Q$, then $a(A_{i,m}.Q^{\bullet n})$ is a summand in the defining equation for $A_{i,j}$ for each $2 \leq i \leq l$.
    4. If $aQ^{\bullet n}$ is a summand in the defining equation for $A_{1,j}$, then $aQ^{\bullet(n+i-1)}$ is a summand in the defining equation for $A_{i,j}$, where $2 \leq i \leq l$.

Example 3. *The process of Example 2 has the level 3; $L_1 = \{X_0, Y_0\}$, $L_2 = \{X_1, Y_1\}$ and $L_3 = \{X_2, Y_2\}$.*

Lemma 4. *Let Q be a non-regular simple process and let Δ be a nBPA process such that $\Delta \| Q \in \mathcal{S}(nBPA)$. Then $\Delta \sim Q^{|\Delta|}$.*

Proof: Let $\Delta \to^* R$ where $|R| = 1$. As Q is non-regular, we can use Lemma 2 and conclude that $\Delta \sim R^{|\Delta|}$. Now it suffices to prove that $R \sim Q$. Let Δ' be a nBPA process such that $\Delta \| Q \sim \Delta'$ and let $m = \max\{|X|,\ X \in \mathit{Var}(\Delta')\}$. As Q is simple and non-regular, $Q \to^* Q^{\bullet m}$ (see Remark 5). Hence $R \| Q^{\bullet m} \sim \alpha$ for some $\alpha \in \mathit{Var}(\Delta')^*$ whose length is at least 2 — thus $\alpha = A.\beta$ for some $\beta \in \mathit{Var}(\Delta')^+$. Let $k = |A|$. Then each two states, which are reachable from $R \| Q^{\bullet m}$ in k norm-decreasing steps are bisimilar—hence $R \| Q^{\bullet m-k} \sim Q^{\bullet m-k+1}$ and from this we have $R \sim Q$. $\square$

Now we can prove the first main theorem of this paper:

Theorem 2. *Let Δ be a non-regular nBPA process and let $\Delta \sim \Delta_1 \| \cdots \| \Delta_n$, where $n \geq 2$, Δ_i is a prime process for each $1 \leq i \leq n$ and Δ_1 is non-regular. Then one of the following possibilities holds:*

- *There is a non-regular simple process Q such that $\Delta \sim Q^{\bullet|\Delta|}$ and $\Delta_i \sim Q$ for each $1 \leq i \leq n$.*
- *There are nBPA processes Δ', Δ'_1 in $DNF(Q)$ such that $\Delta \sim \Delta'$, $\Delta_1 \sim \Delta'_1$, $Level(\Delta') = n$, $Level(\Delta'_1) = 1$ and $\Delta_i \sim Q$ for each $2 \leq i \leq n$.*

Proof: We proceed by induction on n:

- **n=2:** this is an immediate consequence of Proposition 4 and Proposition 5.
- **Induction step:** let $\Delta \sim \Delta_1 \| \cdots \| \Delta_n$. As $\Delta_1 \| \cdots \| \Delta_n \to^* \Delta_1 \| \cdots \| \Delta_{n-1}$, there is a reachable state α of Δ such that $\alpha \sim \Delta_1 \| \cdots \| \Delta_{n-1}$ — hence we can use ind. hypothesis (note that α must be non-regular) and conclude that there are two possibilities:
  1. There is a non-regular simple process Q such that $\Delta_i \sim Q$ for each $1 \leq i \leq n-1$. We prove that $\Delta_n \sim Q$. As $\Delta \sim Q^{n-1} \| \Delta_n$ and $Q^{n-1} \| \Delta_n \to^* Q \| \Delta_n$, we can use Lemma 4 and conclude $\Delta_n \sim Q^{|\Delta_n|}$. Hence $\Delta_n \sim Q$ because Δ_n would not be prime otherwise.
  2. There is a nBPA process Δ'_1 in $DNF(Q)$ such that $\Delta_1 \sim \Delta'_1$, $Level(\Delta'_1) = 1$ and $\Delta_i \sim Q$ for each $1 \leq i \leq n-1$. First we prove that $\Delta_n \sim Q$. As $\Delta_1 \| \Delta_n$ is a reachable state of $\Delta_1 \| \cdots \| \Delta_n$, it belongs to $\mathcal{S}$*(nBPA)*. Let us realize that Δ_n is regular. Assume

the converse—then we can use Proposition 4 and conclude that $\Delta_1 \sim R^{|\Delta_1|}$ for some non-regular simple process R. From this and Remark 5 we can easily prove that $R \sim Q$ and it contradicts regularity of Q.

As Δ_n is regular and $\Delta_1 \| \Delta_2 \in \mathcal{S}(nBPA)$, we can apply Proposition 5; from this (and also from Lemma 3) we get that $\Delta_n \sim Q^{|\Delta_n|}$ and thus $\Delta_n \sim Q$ because Δ_n is prime.

It remains to prove that there is a process Δ' in $DNF(Q)$ such that $Level(\Delta') = n$ and $\Delta \sim \Delta'$. But the process Δ' can be easily constructed by running the algorithm from the proof of Proposition 5 with $\Delta'_1 \| Q^{n-1}$ on input. $\square$

4.2 Decidability results

In this subsection we present several positive decidability results. We show that it is decidable whether a given nBPA process is prime and if the answer is negative, then its decomposition into primes can be effectively constructed. There are also other decidable properties which are summarized in Theorem 4.

Lemma 5. *Let Δ be a nBPA process. It is decidable whether there is a nBPA process Δ' in $DNF(Q)$ such that $\Delta \sim \Delta'$. Moreover, if the answer to the previous question is positive, then the process Δ' can be effectively constructed.*

Proof: We can assume (w.l.o.g.) that Δ is in 3-GNF. If there is a process Δ' in $DNF(Q)$ such that $\Delta \sim \Delta'$, then there is $R \in Var(\Delta)$ such that $R \sim Q$, because Q is a reachable state of Δ'. As Q is a regular simple process, each summand in the def. equation for R must be of the form $a[P]$, where $R \sim P$. As bisimilarity is decidable for nBPA processes, we can construct the set $\mathcal{M}$ of all variables of $Var(\Delta)$ with this property. Each variable from this set is a potential candidate for the variable which is bisimilar to Q (if the set $\mathcal{M}$ is empty, then Δ cannot be bisimilar to any process in $DNF(Q)$).

For each variable $V \in \mathcal{M}$ we now modify the process Δ slightly—we replace each summand of the form aP in the def. equation for V with aV. The resulting process is denoted Δ_V (clearly $\Delta \sim \Delta_V$). For each Δ_V we check whether Δ_V can be transformed into a process in $DNF(V)$. To do this, we first need to realize the following fact: if there is Δ'_V in $DNF(V)$ such that $\Delta_V \sim \Delta'_V$ and $a(A.B)$ is a summand in a def. equation from Δ_V

such that A is non-regular, then $B \sim V^{\bullet|B|}$. It is easy to prove by the technique we already used many times in this paper—as A is non-regular, it can reach a state of an arbitrary norm. Furthermore, there is a reachable state of Δ_V which is of the form $A.B.\gamma$ where $\gamma \in Var(\Delta_V)^*$. We choose sufficiently large α such that $A \to^* \alpha$ and $\alpha.B.\gamma$ must be bisimilar to a state of Δ'_V which is of the form $[Y].V^{\bullet i}$ where $i \geq |B.\gamma|$. From this we get $B \sim V^{\bullet|B|}$.

Now we can describe the promised transformation $\mathcal{T}$ of Δ_V into a process Δ'_V in $DNF(V)$. If this transformation fails, then there is *no* process in $DNF(V)$ bisimilar to Δ_V. $\mathcal{T}$ is invoked on each summand of each def. equation from Δ_V and works as follows:

- $\mathcal{T}(a) = a$
- $\mathcal{T}(aA) = aA$
- $\mathcal{T}(a(A.B)) = aN$ if A is regular. The variable N is the leading variable of $\Delta^{\mathcal{R}}(A)$, whose def. equations are also added to Δ'_V after the following modification: each summand in each def. equation of $\Delta^{\mathcal{R}}(A)$ which is of the form b where $b \in Act$ is replaced with bB.
- $\mathcal{T}(a(A.B)) = a(A.V^{\bullet|B|})$ if A is non-regular and $B \sim V^{\bullet|B|}$. If A is non-regular and $B \not\sim V^{\bullet|B|}$, then $\mathcal{T}$ *fails*.

If there is $V \in \mathcal{M}$ such that $\mathcal{T}$ succeeds for Δ_V, then the process $\Delta'_V \sim \Delta$ is the process we are looking for. Otherwise, there is no process in $DNF(Q)$ bisimilar to Δ. □

Proposition 6. *Let $\Delta_1, \ldots, \Delta_n$, $n \geq 2$ be nBPA processes. It is decidable whether $\Delta_1 \| \cdots \| \Delta_n \in \mathcal{S}(nBPA)$. Moreover, if the answer to the previous question is positive, then a nBPA process Δ such that $\Delta_1 \| \cdots \| \Delta_n \sim \Delta$ can be effectively constructed.*

Proof: By induction on n:

- **n=2:** we distinguish three possibilities (it is decidable which one actually holds—see Proposition 1):
  1. Δ_1 and Δ_2 are regular. Then $\Delta_1 \| \Delta_2 \in \mathcal{S}(nBPA)$ and a bisimilar regular process Δ in normal form can be easily constructed.

2. Δ_1 and Δ_2 are non-regular. Proposition 4 says that there is a non-regular simple process Q such that $\Delta_1 \sim Q^{|\Delta_1|} \sim Q^{\bullet|\Delta_1|}$ and $\Delta_2 \sim Q^{|\Delta_2|} \sim Q^{\bullet|\Delta_2|}$. As Q is a reachable state of $Q^{\bullet|\Delta_2|}$, there is $R \in \mathit{Var}(\Delta_1)$ such that $Q \sim R$. As reachable states of Q are of the form $Q^{\bullet i}$ where $i \in N \cup \{0\}$, each summand $a\alpha$ in the def. equation for R has the property $\alpha \sim R^{\bullet|\alpha|}$. As bisimilarity is decidable for nBPA processes, we can find all variables of $\mathit{Var}(\Delta)$ which have this property—we obtain a set of possible candidates for R (if this set is empty, then $\Delta_1 \| \Delta_2 \notin \mathcal{S}(nBPA)$). Now we check whether the constructed set of candidates contains a variable R such that $\Delta_1 \sim R^{\bullet|\Delta_1|}$. If not, then $\Delta_1 \| \Delta_2 \notin \mathcal{S}(nBPA)$. Otherwise we have R which is bisimilar to Q.

   The same procedure is now applied to Δ_2. If it succeeds, it outputs some $S \in \mathit{Var}(\Delta)$. Now we check whether $R \sim S$. If not, then $\Delta_1 \| \Delta_2 \notin \mathcal{S}(nBPA)$. Otherwise $\Delta_1 \| \Delta_2 \in \mathcal{S}(nBPA)$ and $\Delta_1 \| \Delta_2 \sim R^{\bullet|\Delta_1|+|\Delta_2|}$.

3. Δ_1 is non-regular and Δ_2 is regular (or Δ_1 is regular and Δ_2 is non-regular—this is symmetric). Due to Proposition 5 we know that there is a regular simple process Q and a nBPA process Δ_1' in $DNF(Q)$ such that $\Delta_1 \sim \Delta_1'$ and $\Delta_2 \sim Q^{|\Delta_2|} \sim Q^{\bullet|\Delta_2|}$. An existence of Δ_1' can be checked effectively (see Lemma 5). If it does not exist, then $\Delta_1 \| \Delta_2 \notin \mathcal{S}(nBPA)$. If it exists, it can be also constructed and thus the only thing which remains is to test whether $\Delta_2 \sim Q^{\bullet|\Delta_2|}$. If this test succeeds, then $\Delta_1 \| \Delta_2 \in \mathcal{S}(nBPA)$ and we invoke the algorithm from the proof of Proposition 5 with $\Delta_1' \| Q^{|\Delta_2|}$ on input—it outputs a nBPA process which is bisimilar to $\Delta_1 \| \Delta_2$.

- **Induction step:** if $\Delta_1 \| \cdots \| \Delta_n \in \mathcal{S}(nBPA)$, then also $\Delta_1 \| \cdots \| \Delta_{n-1} \in \mathcal{S}(nBPA)$ and this is decidable by ind. hypothesis—if the answer is negative, then $\Delta_1 \| \cdots \| \Delta_n \notin \mathcal{S}(nBPA)$ and if it is positive, then we can construct a nBPA process Δ' such that $\Delta_1 \| \cdots \| \Delta_{n-1} \sim \Delta'$. Now we check whether $\Delta' \| \Delta_n \in \mathcal{S}(nBPA)$ and construct a bisimilar nBPA process Δ if needed. □

As an immediate consequence of Proposition 6 we get:

Proposition 7. *Let $\Delta, \Delta_1, \ldots, \Delta_n$ be nBPA processes. It is decidable whether $\Delta \sim \Delta_1 \| \cdots \| \Delta_n$.*

Now it is easy to prove the following theorem:

Theorem 3. *Let Δ be a nBPA process. It is decidable whether Δ is prime and if not, its decomposition into primes can be effectively constructed.*

Proof: The technique is the same as in the proof of Theorem 1. We can almost copy the whole proof—the crucial result which allows us to do so is Proposition 7. □

Decidability results which were proved in this subsection are put together by the following theorem:

Theorem 4. *Let $\Delta, \Delta_1, \ldots, \Delta_n$ be nBPA processes. The following problems are decidable:*

- *Is Δ prime? (If not, its decomposition can be effectively constructed)*
- *Is Δ bisimilar to $\Delta_1 \| \cdots \| \Delta_n$?*
- *Does the process $\Delta_1 \| \cdots \| \Delta_n$ belong to $\mathcal{S}$(nBPA)?*
- *Is there any process Δ' such that $\Delta \| \Delta' \in \mathcal{S}$(nBPA)? (if so, an example of such a process can be effectively constructed).*
- *Is there any process Δ' such that $\Delta \sim \Delta_1 \| \cdots \| \Delta_n \| \Delta'$? (if so, Δ' can be effectively constructed).*

4.3 Decidability of bisimilarity for sPA processes

A "structural" way how to construct new processes from older ones is to put them together in parallel. If we do this with nBPA and nBPP processes, we obtain a natural subclass of normed PA processes denoted sPA (**s**imple PA processes):

Definition 8 (sPA processes). *The class of sPA processes is defined as follows:*

$$sPA = \{\Delta_1 \| \cdots \| \Delta_n \mid n \in N,\ \Delta_i \in nBPA \cup nBPP \text{ for each } 1 \leq i \leq n\}$$

The class sPA is strictly greater than the union of nBPA and nBPP processes. This is demonstrated by the following example:

Example 4. *Let Δ_1, Δ_2 be nBPA processes defined as follows:*

$$\Delta_1 : \quad X \stackrel{def}{=} zX + i(Y.X) + q \qquad\qquad \Delta_2 : \quad A \stackrel{def}{=} aA + b(B.A) + r$$
$$\qquad\;\; Y \stackrel{def}{=} i(Y.Y) + d \qquad\qquad\qquad\qquad\quad B \stackrel{def}{=} b(B.B) + c$$

Then there is no *nBPA or nBPP process bisimilar to the sPA process $\Delta_1 \| \Delta_2$. This can be easily proved with the help of pumping lemmas for context-free languages and for languages generated by nBPP processes—see [Chr93].*

Theorem 5. *Let $\Phi = \varphi_1 \| \cdots \| \varphi_n$, $\Psi = \psi_1 \| \cdots \| \psi_m$ be sPA processes. It is decidable whether $\Phi \sim \Psi$.*

Proof: As each φ_i, $1 \leq i \leq n$ and ψ_j, $1 \leq j \leq m$ can be effectively decomposed into a parallel product of primes, we can also construct a decomposition for Φ and Ψ. If $\Phi \sim \Psi$, then those decompositions must be the same up to bisimilarity (see Remark 3). Hence for each prime process η from the decomposition of Φ there must be a prime process ϱ from the decomposition of Ψ such that $\eta \sim \varrho$ (and vice versa). But this can be effectively checked, because bisimilarity is decidable in the union of nBPA and nBPP processes (see Proposition 2). □

5 Conclusions, future work

The main characterization theorem (Theorem 2) says that non-regular nBPA processes which are not prime can be divided into two groups:

1. processes which are bisimilar to a power of some non-regular simple process. It is obvious that each such nBPA process belongs to $\mathcal{S}$*(nBPP)*—see Remark 4.

2. processes which are bisimilar to some process in $DNF(Q)$. It can be proved (with the help of results achieved in [ČKK96]) that each such process does *not* belong to $\mathcal{S}$*(nBPP)*.

From this we can observe that our division based on normal forms corresponds to the membership to $\mathcal{S}$*(nBPP)*.

We have also shown that the decomposition of non-prime nBPA processes can be effectively constructed. This algorithm can be interpreted as a construction of the "most parallel" version of a given sequential program. Finally, we proved that bisimilarity is decidable for sPA processes. (see Definition 8).

The first possible generalization of our results could be the replacement of the '$\|$' operator with the parallel operator of CCS which allows synchronizations on complementary actions. This should not be hard, but we can expect more complicated normal forms. Decidability results should be the same.

A natural question is whether our results can be extended to the class of all (not necessarily normed) BPA processes. The answer is no, because there are quite primitive BPA processes which do not have any decomposition at all—assume e.g., the process $X \stackrel{def}{=} aX$.

Another related open problem is decidability of bisimilarity for normed PA processes. It seems that it should be possible to design at least rich subclasses of normed PA processes where bisimilarity remains decidable. Naturally, we can also ask whether normed PA processes can be effectively decomposed—and this is the area of our future research.

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